Simultaneous dilation and translation tilings of $\mathbb{R}^n$

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Abstract

We solve the wavelet set existence problem. That is, we characterize the full-rank lattices $\Gamma \subset \mathbb{R}^n$ and invertible $n \times n$ matrices A for which there exists a measurable set W such that $\{W + \gamma : \gamma \in \Gamma\}$ and $\{A^j(W) : j \in \mathbb{Z}\}$ are tilings of $\mathbb{R}^n$. The characterization is a non-obvious generalization of the one found by Ionascu and Wang, which solved the problem in the case $n = 2$. As an application of our condition and a theorem of Margulis, we also strengthen a result of Dai, Larson, and Speegle on the existence of wavelet sets by showing that wavelet sets exist for matrix dilations, all of whose eigenvalues λ satisfy $|\lambda| \geq 1$. As another application, we show that the Ionascu-Wang characterization characterizes those dilations whose product of two smallest eigenvalues in absolute value is ≥ 1. The talk is based on a joint work with Darrin Speegle.

Wavelet set problem

Problem (Speegle (1997))

Let A be $n \times n$ invertible matrix and $\Gamma \subset \mathbb{R}^n$ a full rank lattice. For which pairs (A, Γ) does there exist a measurable set $W \subset \mathbb{R}^n$ such that

$$\{A^j(W) : j \in \mathbb{Z}\} \text{ is a measurable tiling of } \mathbb{R}^n \tag{1}$$

and

$$\{W + \gamma : \gamma \in \Gamma\} \text{ is a measurable tiling of } \mathbb{R}^n? \tag{2}$$

A set W that satisfies (1) and (2) is called an (A, Γ) wavelet set.

1D Examples

Example

Let $a = 2$ and $\Gamma = \mathbb{Z}$. Define

$$W = [-1, -1/2] \cup [1/2, 1].$$

Example (Speegle (2003))

Suppose that a dilation factor $a > 1$. Define the set $W =$

$$\left[-\frac{la(a^3-1)}{a^4-1}, -\frac{l(a^3-1)}{a^4-1}\right] \cup \left[\frac{1}{a^4-1}, \frac{l(a-1)}{a^4-1}\right] \cup \left[\frac{la^3(a-1)}{a^4-1}, \frac{a^4}{a^4-1}\right],$$

where $l = \lceil 1/(a-1) \rceil$.

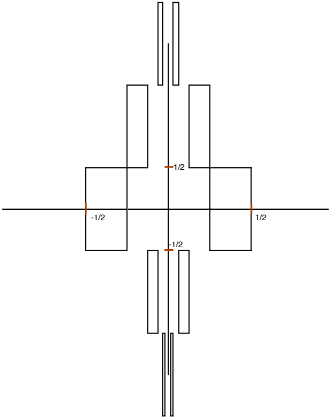
An elementary calculation shows that W is a wavelet set, i.e.,

$$\sum_{j\in\mathbb{Z}} \mathbf{1}_W(a^j\xi) = 1 \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \mathbf{1}_W(\xi + k) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

2D Example

Example (Gu, Speegle (1997))

$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Gamma = \mathbb{Z}^2$. W shown below is (A, Γ) wavelet set.



Motivation 1

$W \subset \mathbb{R}^n$ is a (A, Γ) wavelet set $\iff$ $\psi = \check{1}_W$ is an orthogonal wavelet. That is,

$$\left\{ \psi_{j,k}(x) := |\det B|^{j/2} \, \psi(B^j x + k) : j \in \mathbb{Z}, k \in \Gamma^* \right\},$$

is an orthogonal basis for $L^2(\mathbb{R}^n)$. Here, $B = A^T$ and

$$\Gamma^* = \{x \in \mathbb{R}^n : \langle x, \gamma \rangle \in \mathbb{Z} \quad \forall \gamma \in \Gamma\}.$$

Such wavelets are called minimally supported frequency (MSF). Hernández, Wang, Weiss, *Smoothing minimally supported frequency wavelets* (1996).

Motivation 2

Problem (Steinhaus (1957))

For $\alpha \in SO(n)$, let $\Gamma_\alpha = \alpha\mathbb{Z}^n$. Does there exist a Lebesgue measurable set $E \subset \mathbb{R}^n$, such that E is a fundamental region for each $\mathbb{R}^n/\Gamma_\alpha$?

Kolountzakis and Wolff (1999). False in dimensions 3 and higher.
Open problem in dimension 2.
Jackson and Mauldin (2002). There exists a non-measurable Steinhaus tilings in $\mathbb{R}^2$.

Commonality between Steinhaus problem and wavelet set problem. Multiple actions on $\mathbb{R}^n$ for which there are measurable tilings when considered separately, yet it is not at all clear when and whether there is a single measurable set which tiles by both actions simultaneously.

Tilings by dilations

Theorem (Larson, Schulz, Taylor, Speegle (2006))

Let A be an invertible matrix.

1. *There exists a set that tiles by dilations if and only if A is not orthogonal.*
2. *There exists a set of finite measure that tiles by dilations if and only if $|\det A| \neq 1$.*
3. *There exists a bounded set that tiles by dilations if and only if all (real or complex) eigenvalues of A or A^{-1} have modulus larger than 1.*

Prior Results

Dai, Larson, Speegle (1997). Wavelet sets exist for expanding A, all eigenvalues λ satisfy $|\lambda| > 1$, and Γ any lattice.

Larson, Schulz, Taylor, Speegle (2006). $|\det A| \neq 1$ is necessary for the existence of a wavelet set.

MB, Lemvig (2017). Given any $|\det A| \neq 1$, there exists (A, Γ) wavelet set for almost every lattice Γ.

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Theorem (Ionascu, Wang (2006))

Let A be 2×2 matrix with $|\det A| > 1$ and let Γ be a full rank lattice in $\mathbb{R}^2$. Let λ_1 and λ_2 be the eigenvalues of A such that $|\lambda_1| \geq |\lambda_2|$. There exists an (A, Γ) wavelet set if and only if

1. $|\lambda_2| \geq 1$, *or*
2. $|\lambda_2| < 1$ *and* $\ker(A - \lambda_2 \mathrm{I}) \cap \Gamma = \{0\}$.

2D Example from Guido's office

Example (Speegle (2003))

Let $\Gamma = \mathbb{Z}^2$.

- Wavelet set does not exist for $A = \begin{bmatrix} 3 & 0 \\ \alpha & 1/2 \end{bmatrix}$ for any $\alpha \in \mathbb{R}$.
- Wavelet set exists for $A = \begin{bmatrix} 3 & \alpha \\ 0 & 1/2 \end{bmatrix} \iff \alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem (Ionascu, Wang (2006))

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(i) $|\lambda_2| \geq 1$, *or*

(ii) $|\lambda_2| < 1$ *and* $\ker(A - \lambda_2 I) \cap \Gamma = \{0\}$.

Restatement of Ionescu-Wang characterization

Suppose that the smallest eigenvalue of A is < 1 in modulus.
When dimension $n = 2$ and $|\det A| > 1$ TFAE:

1. for every $R > 0$, $\liminf_{j\to\infty} \#|A^{-j}(\mathrm{B}(0,R)) \cap \Gamma| = 1$,
2. for every $R > 0$, $\liminf_{j\to\infty} \#|A^{-j}(\mathrm{B}(0,R)) \cap \Gamma| < \infty$,
3. for every sublattice $\Lambda \subset \Gamma$, if $V = \mathrm{span}(\Lambda)$ and $d = \dim V$,

$$\liminf_{j\to\infty} m_d(A^{-j}(\mathrm{B}(0,1)) \cap V) < \infty,$$

   where m_d denotes the Lebesgue measure on the subspace V,

4. if V is the space spanned by the eigenvectors associated with eigenvalues less than one in modulus, then $V \cap \Gamma = \{0\}$.

Easy implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) for all n.
KNOWN in higher dimensions: (1) is sufficient, (4) is necessary.
NONE of these characterize the existence of wavelet sets.

Solution of wavelet set problem

Theorem (MB, Speegle (2021))

Let A be an $n \times n$ matrix with $|\det A| > 1$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Then, there exists an (A, Γ) wavelet set $\iff$

$$\sum_{j=1}^{\infty} \frac{1}{\# |A^{-j}(\mathrm{B}(0,1)) \cap \Gamma|} = \infty.$$

Consequences:

- If A is expansive, then $\exists\ J \geq 1$ such that $A^{-j}\,(\mathrm{B}(0,1)) \subset \mathrm{B}(0,1)$ for all $j \geq J \implies$ wavelet set exists.
- (2) $\liminf_{j\to\infty} \#|A^{-j}(\mathrm{B}(0,1)) \cap \Gamma| < \infty$ is sufficient.
- (3) $\liminf_{j\to\infty} m_d(A^{-j}(\mathrm{B}(0,1)) \cap V) < \infty$ is necessary.

Sufficiency

Definition

Let $M \in \mathbb{N}$. The set $U \subset \mathbb{R}^n$ *packs M-redundantly* by A dilations if

$$\sum_{j\in\mathbb{Z}} 1_U(A^j x) \le M \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

$M = 1 \implies U$ *packs* by A dilations.
The set U *covers* by A dilations if

$$\sum_{j\in\mathbb{Z}} 1_U(A^j x) \ge 1 \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

The set U *tiles* by A dilations if it both packs and covers, in which case we call $\{A^j(U) : j \in \mathbb{Z}\}$ a *(measurable) tiling* of $\mathbb{R}^n$.

Similarly, we say a measurable set $V \subset \mathbb{R}^n$ packs M-redundantly, covers, or tiles by Γ translations if $\sum_{\gamma\in\Gamma} 1_V(x+\gamma)$ has the corresponding property.

Elementary packing lemma

Lemma

Let A be an invertible matrix. Let Γ be a full rank lattice in $\mathbb{R}^n$. The following are equivalent:

1. *for every $r > 0$, there exists a sequence $(m_j)_{j\in\mathbb{N}}$ such that:*
   - $\sum 1/m_j = \infty$ *and*
   - $\forall j \in \mathbb{N}$ *the set* $A^{-j}(\mathrm{B}(0,r))$ *packs* m_j *redundantly via* Γ *translations,*
2. $\forall r > 0 \ \sum_{j=1}^{\infty} 1/\#|A^{-j}(\mathrm{B}(0,r)) \cap \Gamma| = \infty$,
3. $\exists r > 0 \ \sum_{j=1}^{\infty} 1/\#|A^{-j}(\mathrm{B}(0,r)) \cap \Gamma| = \infty$.

Difficult packing lemma

Lemma

Let A be an invertible $n \times n$ matrix and Γ a full-rank lattice.
Let W be a set that packs by dilations of A.
Define the dilation equivalency mapping d onto W by

$$d(V) = \bigcup_{j \in \mathbb{Z}} \left(A^j(V) \cap W \right), \qquad \text{where } V \subset \mathbb{R}^n.$$

Suppose that $U \subset W$ is a finite measure set such that:

- $\exists \, (m_j)_{j \in \mathbb{N}} \sum_{j \in \mathbb{N}} 1/m_j = \infty$, *and*
- $\forall j \in \mathbb{N} \; A^{-j}(U)$ *packs m_j redundantly via Γ translations.*

Then, for every $\epsilon > 0$, there exists a set V such that:

1. *V packs via dilations,*
2. *V packs via translations,*
3. $d(V) = U$,
4. $|V| < \epsilon$.

Theorem (Ionescu, Wnag (2006))

Let A be an invertible matrix and let Γ be a full rank lattice in $\mathbb{R}^n$. Suppose there exists a measurable set $U \subset \mathbb{R}^n$ that tiles by A dilations and packs by Γ translations. Then, there exists (A, Γ) wavelet set.

Proof.

Use the Cantor-Schröder-Bernstein theorem. □

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Let A be an invertible matrix and let Γ be a full rank lattice in $\mathbb{R}^n$. Suppose there exists a measurable set $U \subset \mathbb{R}^n$ that tiles by A dilations and packs by Γ translations. Then, there exists (A, Γ) wavelet set.

Proof.

Use the Cantor-Schröder-Bernstein theorem. □

Theorem (Key existence result)

Let A be an invertible $n \times n$ matrix and Γ a full-rank lattice. Suppose there exists a set $W \subset \mathbb{R}^n$ which tiles $\mathbb{R}^n$ by dilations and a partition $(W_m)_{m \in \mathbb{N}}$ of W such that:

- $\forall m \in \mathbb{N}\ \exists (m_j)_{j \in \mathbb{N}}$ *such that* $\sum_{j \in \mathbb{N}} 1/m_j = \infty$ *and*
- $\forall j \in \mathbb{N}\ A^{-j}(W_m)$ *packs* m_j *redundantly via* Γ *translations.*

Then, $\exists\ (A, \Gamma)$ *wavelet set.*

Proof of sufficiency.

By elementary lemma $\sum_{j=1}^{\infty} \frac{1}{\#|A^{-j}(\mathrm{B}(0,1))\cap\Gamma|} = \infty \implies \forall r > 0$

- $\exists (m_j)_{j\in\mathbb{N}}$ such that $\sum 1/m_j = \infty$ and
- $\forall j \in \mathbb{N}$ $A^{-j}(\mathrm{B}(0,r))$ packs m_j redundantly via Γ translations,

Take any set $W \subset \mathbb{R}^n$ that tiles by dilations. The partition $W_m = W \cap \big(\mathrm{B}(0,m) \setminus \mathrm{B}(0,m-1)\big)$, $m \in \mathbb{N}$, fulfills assumptions of the existence theorem. Consequently, $\exists$ (A,Γ) wavelet set. □

Lemma

Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice, and let Ω be a symmetric convex body in $\mathbb{R}^n$. Then,

$$\#|\Omega \cap \Gamma| \geq \frac{1}{2^n |\mathbb{R}^n/\Gamma|} |\Omega|.$$

In, addition if the vectors $\Omega \cap \Gamma$ linearly span $\mathbb{R}^n$, then

$$\#|\Omega \cap \Gamma| \leq \frac{3^n n!}{2^n |\mathbb{R}^n/\Gamma|} |\Omega|.$$

Key Lemma for necessity

Lemma

Let A be an $n \times n$ invertible matrix and let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Suppose that a measurable set $W \subset \mathbb{R}^n$, which packs by Γ translations. Then, $\exists\ C = C(n) > 0$ such that

$$|\mathrm{B}(x,1) \cap A(W)| \le \frac{C}{\#|A^{-1}(\mathrm{B}(0,1)) \cap \Gamma|} \qquad \textit{for all } x \in \mathbb{R}^n.$$

Non-existence of wavelet sets

Proposition (Speegle (2003))

A an invertible $n \times n$ matrix with $|\det A| > 1$, $\Gamma \subset \mathbb{R}^n$ a full-rank lattice, and $W \subset \mathbb{R}^n$. Suppose that there exists $\epsilon > 0$ and an A dilation generator U which contains infinitely many disjoint sets $(V_k)_{k=1}^{\infty}$ of measure at least ϵ such that

$$\sup_{k \in \mathbb{N}} \sum_{j=J}^{\infty} \left| V_k \cap A^j(W) \right| \to 0 \qquad \text{as } J \to \infty.$$

Then, W is not an (A, Γ) wavelet set.

Proof of necessity

Assume that
$$\sum_{j=1}^{\infty} \frac{1}{\#|A^{-j}(\mathrm{B}(0,1)) \cap \Gamma|} < \infty.$$
Suppose $W \subset \mathbb{R}^n$ tiles by Γ translations. There exists an A dilation tiling generator $U \subset \mathbb{R}^n$ which contains a collection of disjoint balls $V_k \subset U$, $k \in \mathbb{N}$, each having the same radius ≤ 1. Let x_k denote the center of V_k. Then, for any $j \in \mathbb{N}$, Key Necessity Lemma applied to the matrix A^j yields
$$|V_k \cap A^j(W)| \leq |\mathrm{B}(x_k,1) \cap A^j(W)| \leq \frac{C}{\#|A^{-j}(\mathrm{B}(0,1)) \cap \Gamma|}.$$
Hence,
$$\sum_{j=J}^{\infty} |V_k \cap A^j(W)| \leq C \sum_{j=J}^{\infty} \frac{1}{\#|A^{-j}(\mathrm{B}(0,1)) \cap \Gamma|} \to 0 \qquad \text{as } J \to \infty.$$
Since $k \in \mathbb{N}$ is arbitrary, Proposition $\implies$ W is not an (A, Γ) wavelet set.

Convenient necessary condition

Theorem

Let A be an $n \times n$ invertible matrix with $|\det A| > 1$.
Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice.
If for any sublattice $\Lambda \subset \Gamma$ we have

$$\lim_{j\to\infty} m_d(V \cap A^{-j}(\mathrm{B}(0,1))) = \infty, \quad \textit{where } V = \operatorname{span}\Lambda,\ d = \dim V,$$

then there is no (A, Γ) wavelet set.

Proof.

An elaborate linear algebra argument shows that

$$\sum_{j=1}^{\infty} \frac{1}{\# |A^{-j}(\mathrm{B}(0,1)) \cap \Gamma|} < \infty. \qquad \square$$

3D Example

Example

Let A be a 3×3 diagonal matrix with entries λ_1, λ_2, λ_3. Let Γ be a full rank lattice in $\mathbb{R}^3$. Suppose that

1. $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$,
2. $|\det A| = |\lambda_1 \lambda_2 \lambda_3| > 1$,
3. $|\lambda_1 \lambda_3| < 1$,
4. there exists $\gamma_1, \gamma_2 \in \Gamma$ such that

$$\text{span}(\gamma_1, \gamma_2) = \text{span}(ae_1 + be_2, e_3), \qquad \text{where } a, b \neq 0.$$

Then, there is no (A, Γ) wavelet set.

Applications

Matrix A is *unipotent* if all of its eigenvalues are 1.

Theorem (Margulis (1971))

Let Γ be a full rank lattice in $\mathbb{R}^n$ and let U_t be a one parameter group of unipotent matrices. There exists $\delta > 0$ such that

$$\sup\{t \in \mathbb{R} : \mathrm{B}(0,\delta) \cap U_t\Gamma = \{0\}\} = \infty.$$

Theorem

Let A be an $n \times n$ matrix such that $|\det A| > 1$ and all eigenvalues of A are ≥ 1 in modulus. Then, for every full rank lattice Γ, there exists an (A, Γ) wavelet set.

Generalization of Ionescu-Wang result

Theorem

Let A be an $n \times n$ diagonal matrix with $|\det A| > 1$ and with eigenvalues arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > 1 > |\lambda_n|$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Assume in addition that

$$|\lambda_n \lambda_{n-1}| \geq 1.$$

Then, there exists an (A, Γ) wavelet set $\iff$ $\Gamma \cap \operatorname{span}(e_n) = \{0\}$, where e_n is the last standard basis vector.

Diophantine approximation

Theorem (Khinchine)

Suppose that ψ *is a positive continuous function on* $(0,\infty)$ *such that* $\psi(j) \to 0$ *as* $j \to \infty$. *There exist numbers* α *and* β, *which together with 1 are linearly independent over* $\mathbb{Z}$, *such that for sufficiently large* j, *there is an integer solution* $(a,b) \in \mathbb{Z}^2$ *of the Diophantine system*

$$\|a\alpha + b\beta\| < \psi(j), \qquad 0 < \max(|a|,|b|) < j,$$

where $\|\cdot\|$ *denotes the distance to the nearest integer.*

Example

There exists a lattice $\Gamma \subset \mathbb{R}^3$ and an invertible, diagonal matrix A with $|\det A| > 1$ such that:

- no (A, Γ) wavelet sets exist and
- $\forall\ R > 0\ \forall F \subset \Gamma$

$$m_d(V \cap A^{-j}(\mathrm{B}(0, R))) \to 0 \qquad \text{as } j \to \infty,$$

  where $V = \operatorname{span} F$ and $d = \dim V$.

For $\psi(j) = 11^{-j}$, let α and β as in Khinchine's Theorem. Let

$$A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Let $\Gamma = \operatorname{span}_{\mathbb{Z}}\{(1, 0, 0), (\alpha, 1, 0), (\beta, 0, 1)\}$.

Open problems

Conjecture

$\exists\ (A, \Gamma)$ *wavelet set* $\iff$ $\exists\ (B, \Gamma^*)$ *orthogonal wavelet.*

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Question (Larson (2007))

Are MSF wavelets minimal with respect to inclusion relation of the Fourier transform support?
Equivalently, does the Fourier support of any orthogonal wavelet contain a wavelet set?

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THANKS FOR YOUR ATTENTION