Simultaneous dilation and translation tilings of $\mathbb{R}^n$

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We solve the wavelet set existence problem. That is, we characterize the full-rank lattices $\Gamma \subset \mathbb{R}^n$ and invertible $n \times n$ matrices $A$ for which there exists a measurable set $W$ such that 

$\{ W + \gamma : \gamma \in \Gamma \}$ and $\{ A^j(W) : j \in \mathbb{Z} \}$ are tilings of $\mathbb{R}^n$. The characterization is a non-obvious generalization of the one found by Ionascu and Wang, which solved the problem in the case $n = 2$. As an application of our condition and a theorem of Margulis, we also strengthen a result of Dai, Larson, and Speegle on the existence of wavelet sets by showing that wavelet sets exist for matrix dilations, all of whose eigenvalues $\lambda$ satisfy $|\lambda| \geq 1$. As another application, we show that the Ionascu-Wang characterization characterizes those dilations whose product of two smallest eigenvalues in absolute value is $\geq 1$. The talk is based on a joint work with Darrin Speegle.
Problem (Speegle (1997))

Let $A$ be $n \times n$ invertible matrix and $\Gamma \subset \mathbb{R}^n$ a full rank lattice. For which pairs $(A, \Gamma)$ does there exist a measurable set $W \subset \mathbb{R}^n$ such that

$$\{A^j(W) : j \in \mathbb{Z}\} \text{ is a measurable tiling of } \mathbb{R}^n$$

(1)

and

$$\{W + \gamma : \gamma \in \Gamma\} \text{ is a measurable tiling of } \mathbb{R}^n?$$

(2)

A set $W$ that satisfies (1) and (2) is called an $(A, \Gamma)$ wavelet set.
Example

Let $a = 2$ and $\Gamma = \mathbb{Z}$. Define

$$W = [-1, -1/2] \cup [1/2, 1].$$

Example (Speegle (2003))

Suppose that a dilation factor $a > 1$. Define the set $W =$

$$\left[ -\frac{la(a^3 - 1)}{a^4 - 1}, -\frac{l(a^3 - 1)}{a^4 - 1} \right] \cup \left[ \frac{1}{a^4 - 1}, \frac{l(a - 1)}{a^4 - 1} \right] \cup \left[ \frac{la^3(a - 1)}{a^4 - 1}, \frac{a^4}{a^4 - 1} \right],$$

where $l = \lceil 1/(a - 1) \rceil$.

An elementary calculation shows that $W$ is a wavelet set, i.e.,

$$\sum_{j \in \mathbb{Z}} 1_W(a^j \xi) = 1 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} 1_W(\xi + k) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}.$$
Example (Gu, Speegle (1997))

\[ A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \Gamma = \mathbb{Z}^2. \text{ W shown below is } (A, \Gamma) \text{ wavelet set.} \]
Motivation 1

$\mathcal{W} \subset \mathbb{R}^n$ is a $(A, \Gamma)$ wavelet set $\iff \psi = \mathbb{1}_\mathcal{W}$ is an orthogonal wavelet. That is,

$$\left\{ \psi_{j,k}(x) := |\det B|^{j/2} \psi(B^j x + k) : j \in \mathbb{Z}, k \in \Gamma^* \right\},$$

is an orthogonal basis for $L^2(\mathbb{R}^n)$. Here, $B = A^T$ and

$$\Gamma^* = \{ x \in \mathbb{R}^n : \langle x, \gamma \rangle \in \mathbb{Z} \ \forall \gamma \in \Gamma \}.$$  

Such wavelets are called minimally supported frequency (MSF).

Problem (Steinhaus (1957))

For $\alpha \in SO(n)$, let $\Gamma_\alpha = \alpha \mathbb{Z}^n$. Does there exist a Lebesgue measurable set $E \subset \mathbb{R}^n$, such that $E$ is a fundamental region for each $\mathbb{R}^n / \Gamma_\alpha$?


Jackson and Mauldin (2002). There exists a non-measurable Steinhaus tilings in $\mathbb{R}^2$.

Commonality between Steinhaus problem and wavelet set problem. Multiple actions on $\mathbb{R}^n$ for which there are measurable tilings when considered separately, yet it is not at all clear when and whether there is a single measurable set which tiles by both actions simultaneously.
Tilings by dilations

Theorem (Larson, Schulz, Taylor, Speegle (2006))

Let $A$ be an invertible matrix.

1. There exists a set that tiles by dilations if and only if $A$ is not orthogonal.

2. There exists a set of finite measure that tiles by dilations if and only if $|\det A| \neq 1$.

3. There exists a bounded set that tiles by dilations if and only if all (real or complex) eigenvalues of $A$ or $A^{-1}$ have modulus larger than 1.
Prior Results

Dai, Larson, Speegle (1997). Wavelet sets exist for expanding $A$, all eigenvalues $\lambda$ satisfy $|\lambda| > 1$, and $\Gamma$ any lattice.

Larson, Schulz, Taylor, Speegle (2006). $|\det A| \neq 1$ is necessary for the existence of a wavelet set.

MB, Lemvig (2017). Given any $|\det A| \neq 1$, there exists $(A, \Gamma)$ wavelet set for almost every lattice $\Gamma$. 
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**Theorem (Ionascu, Wang (2006))**

Let $A$ be $2 \times 2$ matrix with $|\det A| > 1$ and let $\Gamma$ be a full rank lattice in $\mathbb{R}^2$. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $A$ such that $|\lambda_1| \geq |\lambda_2|$. There exists an $(A, \Gamma)$ wavelet set if and only if

(i) $|\lambda_2| \geq 1$, or

(ii) $|\lambda_2| < 1$ and $\ker(A - \lambda_2 I) \cap \Gamma = \{0\}$. 
Example (Speegle (2003))

Let $\Gamma = \mathbb{Z}^2$.

- Wavelet set does not exist for $A = \begin{bmatrix} 3 & 0 \\ \alpha & 1/2 \end{bmatrix}$ for any $\alpha \in \mathbb{R}$.
- Wavelet set exists for $A = \begin{bmatrix} 3 & \alpha \\ 0 & 1/2 \end{bmatrix} \iff \alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem (Ionascu, Wang (2006))

Let $A$ be $2 \times 2$ matrix with $|\det A| > 1$ and let $\Gamma$ be a full rank lattice in $\mathbb{R}^2$. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $A$ such that $|\lambda_1| \geq |\lambda_2|$. There exists an $(A, \Gamma)$ wavelet set if and only if

- (i) $|\lambda_2| \geq 1$, or
- (ii) $|\lambda_2| < 1$ and $\ker(A - \lambda_2 I) \cap \Gamma = \{0\}$. 
Restatement of Ionescu-Wang characterization

Suppose that the smallest eigenvalue of $A$ is $< 1$ in modulus. When dimension $n = 2$ and $|\det A| > 1$ TFAE:

1. for every $R > 0$, $\liminf_{j\to\infty} \# |A^{-j}(B(0, R)) \cap \Gamma| = 1$,
2. for every $R > 0$, $\liminf_{j\to\infty} \# |A^{-j}(B(0, R)) \cap \Gamma| < \infty$,
3. for every sublattice $\Lambda \subset \Gamma$, if $V = \text{span}(\Lambda)$ and $d = \dim V$,
   $$\liminf_{j\to\infty} m_d(A^{-j}(B(0, 1)) \cap V) < \infty,$$
   where $m_d$ denotes the Lebesgue measure on the subspace $V$,
4. if $V$ is the space spanned by the eigenvectors associated with eigenvalues less than one in modulus, then $V \cap \Gamma = \{0\}$.

Easy implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) for all $n$.

KNOWN in higher dimensions: (1) is sufficient, (4) is necessary.
NONE of these characterize the existence of wavelet sets.

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Theorem (MB, Speegle (2021))

Let $A$ be an $n \times n$ matrix with $|\det A| > 1$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Then, there exists an $(A, \Gamma)$ wavelet set $\iff$

$$\sum_{j=1}^{\infty} \frac{1}{\# |A^{-j}(B(0,1)) \cap \Gamma|} = \infty.$$ 

Consequences:

- If $A$ is expansive, then $\exists J \geq 1$ such that $A^{-j} (B(0,1)) \subset B(0,1)$ for all $j \geq J \implies$ wavelet set exists.
- (2) $\liminf_{j \to \infty} \# |A^{-j}(B(0,1)) \cap \Gamma| < \infty$ is sufficient.
- (3) $\liminf_{j \to \infty} m_d(A^{-j}(B(0,1)) \cap V) < \infty$ is necessary.
Let $M \in \mathbb{N}$. The set $U \subset \mathbb{R}^n$ packs $M$-redundantly by $A$ dilations if

$$\sum_{j \in \mathbb{Z}} 1_U(A^j x) \leq M \quad \text{for a.e. } x \in \mathbb{R}^n.$$ 

$M = 1 \implies U$ packs by $A$ dilations.

The set $U$ covers by $A$ dilations if

$$\sum_{j \in \mathbb{Z}} 1_U(A^j x) \geq 1 \quad \text{for a.e. } x \in \mathbb{R}^n.$$ 

The set $U$ tiles by $A$ dilations if it both packs and covers, in which case we call $\{A^j(U) : j \in \mathbb{Z}\}$ a (measurable) tiling of $\mathbb{R}^n$.

Similarly, we say a measurable set $V \subset \mathbb{R}^n$ packs $M$-redundantly, covers, or tiles by $\Gamma$ translations if $\sum_{\gamma \in \Gamma} 1_V(x + \gamma)$ has the corresponding property.
Let $A$ be an invertible matrix. Let $\Gamma$ be a full rank lattice in $\mathbb{R}^n$. The following are equivalent:

1. for every $r > 0$, there exists a sequence $(m_j)_{j \in \mathbb{N}}$ such that:
   - $\sum 1/m_j = \infty$ and
   - $\forall j \in \mathbb{N}$ the set $A^{-j}(B(0, r))$ packs $m_j$ redundantly via $\Gamma$ translations,

2. $\forall r > 0 \sum_{j=1}^{\infty} 1/\#|A^{-j}(B(0, r)) \cap \Gamma| = \infty$,

3. $\exists r > 0 \sum_{j=1}^{\infty} 1/\#|A^{-j}(B(0, r)) \cap \Gamma| = \infty$. 

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Lemma

Let $A$ be an invertible $n \times n$ matrix and $\Gamma$ a full-rank lattice. Let $W$ be a set that packs by dilations of $A$. Define the dilation equivalency mapping $d$ onto $W$ by

$$d(V) = \bigcup_{j \in \mathbb{Z}} (A^j(V) \cap W),$$

where $V \subset \mathbb{R}^n$.

Suppose that $U \subset W$ is a finite measure set such that:

1. $\exists (m_j)_{j \in \mathbb{N}} \sum_{j \in \mathbb{N}} 1/m_j = \infty$, and
2. $\forall j \in \mathbb{N} \ A^{-j}(U)$ packs $m_j$ redundantly via $\Gamma$ translations.

Then, for every $\epsilon > 0$, there exists a set $V$ such that:

1. $V$ packs via dilations,
2. $V$ packs via translations,
3. $d(V) = U$,
4. $|V| < \epsilon$. 

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Theorem (Ionescu, Wnag (2006))

Let $A$ be an invertible matrix and let $\Gamma$ be a full rank lattice in $\mathbb{R}^n$. Suppose there exists a measurable set $U \subset \mathbb{R}^n$ that tiles by $A$ dilations and packs by $\Gamma$ translations. Then, there exists $(A, \Gamma)$ wavelet set.

Proof.

Use the Cantor-Schröder-Bernstein theorem.
Theorem (Ionescu, Wang (2006))

Let $A$ be an invertible matrix and let $\Gamma$ be a full rank lattice in $\mathbb{R}^n$. Suppose there exists a measurable set $U \subset \mathbb{R}^n$ that tiles by $A$ dilations and packs by $\Gamma$ translations. Then, there exists $(A, \Gamma)$ wavelet set.

Proof.

Use the Cantor-Schröder-Bernstein theorem.

Theorem (Key existence result)

Let $A$ be an invertible $n \times n$ matrix and $\Gamma$ a full-rank lattice. Suppose there exists a set $W \subset \mathbb{R}^n$ which tiles $\mathbb{R}^n$ by dilations and a partition $(W_m)_{m \in \mathbb{N}}$ of $W$ such that:

- $\forall m \in \mathbb{N} \exists (m_j)_{j \in \mathbb{N}}$ such that $\sum_{j \in \mathbb{N}} 1/m_j = \infty$ and
- $\forall j \in \mathbb{N} \ A^{-j}(W_m)$ packs $m_j$ redundantly via $\Gamma$ translations.

Then, $\exists (A, \Gamma)$ wavelet set.
Proof of sufficiency.

By elementary lemma \( \sum_{j=1}^{\infty} \frac{1}{\#|A^{-j}(B(0,1))\cap\Gamma|} = \infty \implies \forall r > 0 \)

- \( \exists (m_j)_{j \in \mathbb{N}} \) such that \( \sum 1/m_j = \infty \) and
- \( \forall j \in \mathbb{N} \ A^{-j}(B(0, r)) \) packs \( m_j \) redundantly via \( \Gamma \) translations,

Take any set \( W \subset \mathbb{R}^n \) that tiles by dilations. The partition \( W_m = W \cap (B(0, m) \setminus B(0, m - 1)) \), \( m \in \mathbb{N} \), fulfills assumptions of the existence theorem. Consequently, \( \exists (A, \Gamma) \) wavelet set. \( \square \)
Lemma

Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice, and let $\Omega$ be a symmetric convex body in $\mathbb{R}^n$. Then,

$$\#|\Omega \cap \Gamma| \geq \frac{1}{2^n|\mathbb{R}^n/\Gamma|} |\Omega|.$$ 

In addition if the vectors $\Omega \cap \Gamma$ linearly span $\mathbb{R}^n$, then

$$\#|\Omega \cap \Gamma| \leq \frac{3^n n!}{2^n|\mathbb{R}^n/\Gamma|} |\Omega|.$$
Lemma

Let $A$ be an $n \times n$ invertible matrix and let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Suppose that a measurable set $W \subset \mathbb{R}^n$, which packs by $\Gamma$ translations. Then, $\exists C = C(n) > 0$ such that

$$|B(x, 1) \cap A(W)| \leq \frac{C}{\#|A^{-1}(B(0, 1)) \cap \Gamma|}$$

for all $x \in \mathbb{R}^n$. 

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Proposition (Speegle (2003))

A an invertible \( n \times n \) matrix with \(|\det A| > 1\), \( \Gamma \subset \mathbb{R}^n \) a full-rank lattice, and \( W \subset \mathbb{R}^n \). Suppose that there exists \( \epsilon > 0 \) and an \( A \) dilation generator \( U \) which contains infinitely many disjoint sets \((V_k)_{k=1}^\infty\) of measure at least \( \epsilon \) such that

\[
\sup_{k \in \mathbb{N}} \sum_{j=J}^\infty |V_k \cap A^j(W)| \to 0 \quad \text{as } J \to \infty.
\]

Then, \( W \) is not an \((A, \Gamma)\) wavelet set.
Proof of necessity

Assume that
\[ \sum_{j=1}^{\infty} \frac{1}{\# |A^{-j}(B(0, 1)) \cap \Gamma|} < \infty. \]

Suppose \( W \subset \mathbb{R}^n \) tiles by \( \Gamma \) translations. There exists an \( A \) dilation tiling generator \( U \subset \mathbb{R}^n \) which contains a collection of disjoint balls \( V_k \subset U, k \in \mathbb{N} \), each having the same radius \( \leq 1 \). Let \( x_k \) denote the center of \( V_k \). Then, for any \( j \in \mathbb{N} \), Key Necessity Lemma applied to the matrix \( A^j \) yields
\[ |V_k \cap A^j(W)| \leq |B(x_k, 1) \cap A^j(W)| \leq \frac{C}{\# |A^{-j}(B(0, 1)) \cap \Gamma|}. \]

Hence,
\[ \sum_{j=J}^{\infty} |V_k \cap A^j(W)| \leq C \sum_{j=J}^{\infty} \frac{1}{\# |A^{-j}(B(0, 1)) \cap \Gamma|} \to 0 \quad \text{as} \quad J \to \infty. \]

Since \( k \in \mathbb{N} \) is arbitrary, Proposition \( \implies \) \( W \) is not an \( (A, \Gamma) \) wavelet set.
Theorem

Let $A$ be an $n \times n$ invertible matrix with $|\det A| > 1$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. If for any sublattice $\Lambda \subset \Gamma$ we have

$$\lim_{j \to \infty} m_d(V \cap A^{-j}(B(0, 1))) = \infty,$$

where $V = \text{span} \, \Lambda$, $d = \dim V$, then there is no $(A, \Gamma)$ wavelet set.

Proof.

An elaborate linear algebra argument shows that

$$\sum_{j=1}^{\infty} \frac{1}{\# |A^{-j}(B(0, 1)) \cap \Gamma|} < \infty.$$
Let $A$ be a $3 \times 3$ diagonal matrix with entries $\lambda_1$, $\lambda_2$, $\lambda_3$. Let $\Gamma$ be a full rank lattice in $\mathbb{R}^3$. Suppose that

1. $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$, 
2. $|\text{det} \ A| = |\lambda_1 \lambda_2 \lambda_3| > 1$, 
3. $|\lambda_1 \lambda_3| < 1$, 
4. there exists $\gamma_1, \gamma_2 \in \Gamma$ such that

$$\text{span}(\gamma_1, \gamma_2) = \text{span}(ae_1 + be_2, e_3), \quad \text{where } a, b \neq 0.$$ 

Then, there is no $(A, \Gamma)$ wavelet set.
Matrix $A$ is \textit{unipotent} if all of its eigenvalues are 1.

**Theorem (Margulis (1971))**

Let $\Gamma$ be a full rank lattice in $\mathbb{R}^n$ and let $U_t$ be a one parameter group of unipotent matrices. There exists $\delta > 0$ such that

$$\sup\{ t \in \mathbb{R} : B(0, \delta) \cap U_t \Gamma = \{0\} \} = \infty.$$ 

**Theorem**

Let $A$ be an $n \times n$ matrix such that $|\det A| > 1$ and all eigenvalues of $A$ are $\geq 1$ in modulus. Then, for every full rank lattice $\Gamma$, there exists an $(A, \Gamma)$ wavelet set.
Theorem

Let $A$ be an $n \times n$ diagonal matrix with $|\det A| > 1$ and with eigenvalues arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > 1 > |\lambda_n|$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Assume in addition that $|\lambda_n \lambda_{n-1}| \geq 1$.

Then, there exists an $(A, \Gamma)$ wavelet set $\iff \Gamma \cap \text{span}(e_n) = \{0\}$, where $e_n$ is the last standard basis vector.
Theorem (Khinchine)

Suppose that $\psi$ is a positive continuous function on $(0, \infty)$ such that $\psi(j) \to 0$ as $j \to \infty$. There exist numbers $\alpha$ and $\beta$, which together with 1 are linearly independent over $\mathbb{Z}$, such that for sufficiently large $j$, there is an integer solution $(a, b) \in \mathbb{Z}^2$ of the Diophantine system

$$\|a\alpha + b\beta\| < \psi(j), \quad 0 < \max(|a|, |b|) < j,$$

where $\| \cdot \|$ denotes the distance to the nearest integer.
Example

There exists a lattice $\Gamma \subset \mathbb{R}^3$ and an invertible, diagonal matrix $A$ with $|\det A| > 1$ such that:
- no $(A, \Gamma)$ wavelet sets exist and
- $\forall \ R > 0 \ \forall F \subset \Gamma$

$$m_d(\mathcal{V} \cap A^{-j}(B(0, R))) \to 0 \quad \text{as } j \to \infty,$$

where $\mathcal{V} = \text{span } F$ and $d = \text{dim } \mathcal{V}$.

For $\psi(j) = 11^{-j}$, let $\alpha$ and $\beta$ as in Khinchine’s Theorem. Let

$$A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Let $\Gamma = \text{span}_{\mathbb{Z}}\{(1, 0, 0), (\alpha, 1, 0), (\beta, 0, 1)\}$. 
Conjecture

\[ \exists (A, \Gamma) \text{ wavelet set} \iff \exists (B, \Gamma^*) \text{ orthogonal wavelet.} \]
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\[ \exists (A, \Gamma) \text{ wavelet set } \iff \exists (B, \Gamma^*) \text{ orthogonal wavelet.} \]

Question (Larson (2007))

Are MSF wavelets minimal with respect to inclusion relation of the Fourier transform support?

Equivalently, does the Fourier support of any orthogonal wavelet contain a wavelet set?
Open problems

Conjecture

\[ \exists (A, \Gamma) \text{ wavelet set } \iff \exists (B, \Gamma^*) \text{ orthogonal wavelet. } \]

Question (Larson (2007))

Are MSF wavelets minimal with respect to inclusion relation of the Fourier transform support?
Equivalently, does the Fourier support of any orthogonal wavelet contain a wavelet set?

THANKS FOR YOUR ATTENTION