Singular Integrals and Wevelet–Type Decomposition with Aplications to Littlewood-Paley Theory and Hardy Spaces in Dunkl Setting

Yongsheng Han
Auburn University
(Joint work with Ya Han, M.-Y. Lee, and J. Li and CQ. Tan)

October 8, 2022

(X, ρ, μ) is said to be space of homogeneous type Coifman and Weiss if for all $x, y, z \in X$, ρ satisfies

(i) $\rho(x, y) \geq 0$ and $rho(x, y) = 0$ iff $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$;

(iii) $\rho(x, y) \leq A\{\rho(x, z) + \rho(z, y)\}$ with $A \geq 1$.

And the nonegative measure μ satisfies the doubling condition: $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X$ and $r > 0$ with $B(x, r) = \{y : y \in X, \rho(x, y) < r\}$.

(X, ρ, μ) is Ahlfors's space if for all $x \in X, r > 0$ and $0 < \theta < 1$,

$$|\rho(x, y) - \rho(x', y)| \leq C(\rho(x, x'))^\theta [\rho(x, y) + \rho(x'y)]^{1-\theta};$$

$$C^{-1} r\mu(B(x, r)) \leq Cr.$$

(X, ρ, μ) is an RD space if for all $x \in X, r > 0$ and $0 < \theta < 1$,

$$|\rho(x, y) - \rho(x', y)| \leq C(\rho(x, x'))^\theta [\rho(x, y) + \rho(x'y)]^{1-\theta};$$

$$c\lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C\lambda^k \mu(B(x, r))'$$

for all $x \in X$, $0 < r < \sup_{x,y \in X} \rho(x, y)/2$ and

$1 \leqslant \lambda < \sup_{x,y \in X} \rho(x, y)/2r$.

$(\mathbb{R}^N, \|\cdot\|, \omega)$ is an RD space of homogeneous type of Coifman and Weiss. Let $C_0^\eta(\mathbb{R}^N)$ denote the space of continuous functions f with compact support and

$$\|f\|_\eta := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|^\eta} < \infty.$$

Definition

An operator $T: C_0^\eta(\mathbb{R}^N) \to (C_0^\eta(\mathbb{R}^N))'$ is said to be a Calderón-Zygmund singular integral operator if $K(x,y)$, the kernel of T, satisfies the following estimates: for some $0 < \varepsilon \le 1$,

(i) $|K(x,y)| \le \frac{C}{\omega(x,\|x-y\|)}, x \ne y$;

(ii) $|K(x,y) - K(x',y)|, |K(y,x) - K(y,x')| \le \frac{C}{\omega(x,\|x-y\|)} \left(\frac{\|x-x'|}{\|x-y\|}\right)^\varepsilon$, for $\|x-x'\| \le \frac{1}{2}\|x-y\|$.

(iii) $\langle T(f), g\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y) f(x) g(y) d\omega(x) d\omega(y)$, where $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset, \omega(x,\|x-y\|) = \omega(B(x,\|x-y\|))$.

Theorem

Suppose that T is a Calderón-Zygmund singular integral operator. Then T extends to a bounded operator on $L^2(\mathbb{R}^N, \omega)$ if and only if

(a) $T(1)(x) \in BMO(\mathbb{R}^N, \omega)$;

(b) $T^*(1)(x) \in BMO(\mathbb{R}^N, \omega)$;

(c) *T has WBP:if there exist $\eta > 0$ and $C < \infty$ such that*

$$|\langle K, f\rangle| \leq C \max\{\omega(B(x_0, r)), \omega(B(y_0, r))\}$$

for all $f \in C_0^\eta(\mathbb{R}^N \times \mathbb{R}^N)$ with supp$(f) \subseteq B(x_0, r) \times B(y_0, r)$,$x_0, y_0 \in \mathbb{R}^N$,$\|f\|_\infty \leq 1$, $\|f(\cdot, y)\|_\eta \leq r^{-\eta}$ for all $y \in \mathbb{R}^N$ and $\|f(x, \cdot)\|_\eta \leq r^{-\eta}$ for all $x \in \mathbb{R}^N$.

Coifman's approximation to the identity:

$$S_k = M_k T_k W_k T_k M_k$$

$\theta : \mathbb{R} \mapsto [0,1]$ be a smooth function which is 1 for $\|x\| \le 1$ and vanishes for $\|x\| \ge 2$.

$$T_k(f)(x) = \int_{\mathbb{R}^N} \theta(2^k\|x-y\|)f(y)d\omega(y), \quad k \in \mathbb{Z}.$$

$$M_k(x) := \frac{1}{T_k(1)(x)}$$

$$W_k(x) := \left[T_k\left(\frac{1}{T_k(1)}\right)(x)\right]^{-1}$$

.

$S_k(x,y)$ satisfy:

(i) $S_k(x,y) = S_k(y,x)$;

(ii) $S_k(x,y) = 0$ if
$\|x-y\| > 2^{4-k}$and$|S_k(x,y)| \le \dfrac{C}{V_k(x)+V_k(y)}$;

(iii) $|S_k(x,y) - S_k(x',y)| \le C\dfrac{2^k\|x-x'\|}{V_k(x)+V_k(y)}$ for
$\|x-x'\| \le 2^{8-k}$;

(iv) $|S_k(x,y) - S_k(x,y')| \le C\dfrac{2^k\|y-y'\|}{V_k(x)+V_k(y)}$ for
$\|y-y'\| \le 2^{8-k}$;

(v) $\left|[S_k(x,y) - S_k(x',y)] - [S_k(x,y') - S_k(x',y')]\right| \leq C\dfrac{2^k\|x-x'\|2^k\|y-y'\|}{V_k(x)+V_k(y)}$ for $\|x-x'\| \leq 2^{8-k}$ and $\|y-y'\| \leq 2^{8-k}$;

(vi) $$\int_{\mathbb{R}^N} S_k(x,y)d\omega(x) = 1 \qquad \text{for all } y \in \mathbb{R}^N;$$

(vii) $$\int_{\mathbb{R}^N} S_k(x,y)d\omega(y) = 1 \qquad \text{for all } x \in \mathbb{R}^N.$$

$V_k(x) = \omega(B(x, 2^{-k}))$ for $x \in \mathbb{R}^N$.

Let $D_k = S_{k+1} - S_k$. Coifman's decomposition of the identity on $L^2(\mathbb{R}^N, \omega)$:

$$I = \sum_{k=-\infty}^{\infty} D_k = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} D_k D_j = T_M + R_M,$$

where

$$T_M = \sum_{\{j,k \in \mathbb{Z}:\, |k-j| \le M\}} D_k D_j = \sum_{k \in \mathbb{Z}} D_k D_k^M$$

$$D_k^M = \sum_{|j| \le M} D_{k+j}$$

$$R_M = \sum_{\{j,k \in \mathbb{Z}:\, |k-j| > M\}} D_k D_j.$$

$|D_jD_k(x,y)| \leq C2^{-|j-k|}\frac{1}{V_{j\wedge k}(x)+V_{j\wedge k}(y)}$, where $j \wedge k = \min\{j,k\}$.

$$\|D_jD_k\|_{L^2(\omega)\mapsto L^2(\omega)} \lesssim 2^{-|j-k|}.$$

By the Cotlar-Stein Lemma,

$$\|R_M(f)\|_{L^2(\omega)} \leq C2^{-M}\|f\|_{L^2(\omega)}$$

Fix a large M, T_M^{-1}, the inverse of T_M, is bounded on $L^2(\mathbb{R}^N,\omega)$.

$$I = T_M^{-1}T_M = \sum_{k\in\mathbb{Z}} T_M^{-1}D_k^M D_k = T_M T_M^{-1} = \sum_{k\in\mathbb{Z}} D_k^M D_k T_M^{-1},$$

in $L^2(\mathbb{R}^N,\omega)$.

Almost orthogonal estimate: if $T(1) = T^*(1) = 0$ and $T \in WBP$:

$$\|D_k^* T D_{k'}\|_{L^2(\omega) \mapsto L^2(\omega)} \lesssim 2^{-|k-k'|}.$$

Denote

$$U_{L_1,L_2} = \sum_{L_1 \le k \le L_2} D_k D_k^M.$$

$$|\langle U_{L_1,L_2} f_1, T U_{L_1',L_2'} f_2 \rangle| \lesssim \|f_1\|_{L^2(d\omega)} \|f_2\|_{L^2(d\omega)}.$$

Denote $\{Q^k\}$ are all dyadic cubes in $\mathbb{R}^N$ with the length 2^{-k-M}. Let $I = T_M + R_M$, where $T_M(f)(x) = \sum_{k\in\mathbb{Z}} D_k^M D_k(f)(x) d\omega(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q^k} \omega(Q) D_k^M(x, x_Q) D_k(f)(x_Q) + R^M(f)(x) = \widetilde{T}_M(f)(x) + R^M(f)(x)$,

$\widetilde{T}_M(f)(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q^k} \omega(Q) D_k^M(x, x_Q) D_k(f)(x_Q)$ and

$R^M(f)(x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^N} \sum_{Q\in Q^k} \int_Q [D_k^M(x, y) D_k(f)(y) - D_k^M(x, x_Q) D_k(f)(x_Q)] d\omega(y)$.

$$I = \widetilde{T}_M(f)(x) + R^M(f)(x) + R_M(f)(x).$$

Definition

A function $f(x)$ defined on $\mathbb{R}^N$ is said to be a test function if there exits a constant C such that for $0 < \beta \le 1, \gamma > 0, r > 0$ and $x_0 \in \mathbb{R}^N$,

(i) $|f(x)| \le \frac{C}{V(x,r+\|x-x_0\|)}\left(\frac{r}{r+\|x-x_0\|}\right)^{\gamma}$;

(ii) $|f(x) - f(x')| \le$
$$C\left(\frac{\|x-x'\|}{r+\|x-x_0\|}\right)^{\beta}\frac{1}{V(x,r+\|x-x_0\|)}\left(\frac{r}{r+\|x-x_0\|}\right)^{\gamma},$$
for $\|x-x'\| \le \frac{1}{2}(r+\|x-x_0\|)$;

(iii) $\displaystyle\int_{\mathbb{R}^N} f(x)d\omega(x) = 0$.

We denote such a test function by $f \in \mathcal{M}(\beta,\gamma,r,x_0)$ and $\|f\|_{\mathcal{M}(\beta,\gamma,r,x_0)}$, the norm in $\mathcal{M}(\beta,\gamma,r,x_0)$, is defined by the smallest C satisfying the above conditions (i) and (ii).

Theorem

Suppose that T is a Calderón-Zygmund operator with $T(1) = T^*(1) = 0$, $T \in WBP$ and $K(x,y)$, the kernel of T, satisfies the additional second order smoothness condition:

$$|[K(x,y) - K(x',y)] - [K(x,y') - K(x',y')]|$$

$$\leqslant C\Big(\frac{\|x-x'\|\|y-y'\|}{\|x-y\|^2}\Big)^{\varepsilon}\frac{C}{\omega(B(x,\|x-y\|))}$$

for $\|x-x'\| \leqslant \frac{1}{2}\|x-y\|$ and $\|y-y'\| \leqslant \frac{1}{2}\|x-y\|$. Then for fixed $r > 0, x_0 \in \mathbb{R}^N$, and $0 < \beta, \gamma < \varepsilon$,

$$\|T(f)\|_{\mathcal{M}(\beta,\gamma,r,x_0)} \leqslant C\|f\|_{\mathcal{M}(\beta,\gamma,r,x_0)}.$$

Wevelet–Type Decomposition on Coifman and Weiss's Space of Homogeneous Type

$$\|R^M(f)(x) + R_M(f)(x)\|_{\mathcal{M}(\beta,\gamma,r,x_0)} \leqslant C2^{-M}\|f\|_{\mathcal{M}(\beta,\gamma,r,x_0)}.$$

$$I = \widetilde{T}_M(f)(x) + R^M(f)(x) + R_M(f)(x).$$

$$f(x) = (\widetilde{T}_M)^{-1}\widetilde{T}_M(f)(x) = \widetilde{T}_M(\widetilde{T}_M)^{-1}(f)(x) =$$

$$\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k}\omega(Q)\big((\widetilde{T}_M)^{-1}(D_k^M(\cdot,x_Q))\big)(x)D_k(f)(x_Q) =$$

$$\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k}\omega(Q)D_k(x,x_Q)\big((D_k^M(\cdot,x_Q))(\widetilde{T}_M)^{-1}\big)(f)(x_Q)$$

Theorem

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q^k} \omega(Q)\widetilde{D}_k(x, x_Q)D_k(f)(x_Q)$$

$$= \sum_{k\in\mathbb{Z}} \sum_{Q\in Q^k} \omega(Q)D_k(x, x_Q)\widetilde{\widetilde{D}}_k(f)(x_Q)$$

where the series converge in $L^p(\omega), 1 < p < \infty, \mathcal{M}(\beta, \gamma, r, x_0)$, *and in* $(\mathcal{M}(\beta, \gamma, r, x_0))'$, *the dual of in* $\mathcal{M}(\beta, \gamma, r, x_0)$, *and moreover, the kernels of the operators* $\widetilde{D}_k$ *satisfy the the following conditions:*

(i) $|\widetilde{D}_k(x,y)| \leq C \frac{1}{V_k(x)+V_k(y)+V(x,y)} \frac{2^{-k}}{2^{-k}+\|x-y\|}$;

(ii) $|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \leq$
$C \times \frac{\|x-x'\|}{2^{-k}+\|x-x'\|} \frac{1}{V_k(x)+V_k(y)+V(x,y)} \frac{2^{-k}}{2^{-k}+\|x-y\|}$ for
$\|x-x'\| \leq (2^{-k}+\|x-y\|)/2$;

(iii) $\int_{\mathbb{R}^N} \widetilde{D}_k(x,y) d\omega(x) = 0$;

(iv) $\int_{\mathbb{R}^N} \widetilde{D}_k(x,y) d\omega(y) = 0$.

(v) $\widetilde{\widetilde{D}}_k(x,y)$ satisfy the above conditions with x and y interchanged.

Definition

Suppose that $f \in (\mathcal{M}(\beta, \gamma, r, x_0))'$. $S_{cw}(f)$, the Littlewood-Paley square function of f for space of homogeneous type $(\mathbb{R}^N, \|\cdot\|, \omega)$, is defined by

$$S_{cw}(f)(x) = \left\{ \sum_{k=-\infty}^{\infty} \sum_{Q \in Q^k} |D_k f(x_Q)|^2 \chi_Q(x) \right\}^{1/2}.$$

Definition

The Hardy space $H^p_{cw}, p \leq 1$, is the collection of all $f \in (\mathcal{M}(\beta, \gamma, r, x_0))'$. $S_{cw}(f)$ such that $\|S_{cw}(f)\|_p < \infty$. If $f \in H^p_{cw}$ the norm of f is defined by

$$\|f\|_{H^p_{cw}} =: \|S_{cw}(f)\|_p.$$

(X, ρ, μ) with ρ has no any regularity and μ satisfies the doubling condition only,

Adapting the developed randomized dyadic structure on space of homogeneous type (X, ρ, μ) where ρ is the quasi-metric without any regularity and the measure μ satisfies the doubling condition only, Auscher and Hytönen builded a remarkable orthonormal basis of Hölder-continuos wavelet with exponential decay. Using this wavelet basis they provided the $T1$ theorem in this general setting. Applying Auscher-Hytönen's orthonormal basis, the Hardy space and the product Hardy space were developed.

The Dunkl setting can be considered as an RD space of homogeneous type of Coifman and Weiss by $(\mathbb{R}^{\mathbb{N}}, ||\cdot||, d\omega)$ where $||\cdot||$ is the Euclidean metric and the Dunkl measure $d\omega$ is defined by

$$d\omega(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{\kappa(\alpha)} dx,$$

where the multiplicity function κ defined on R (invariant under G) is fixed ≥ 0. Moreover, $d\omega(x)$ satisfies the doubling and reverse doubling properties, that is, there is a constant $C > 0$ such that for all $x \in \mathbb{R}^N, r > 0, \lambda \geqslant 1$ and $\mathbf{N} = N + \sum_{\alpha \in R} \kappa(\alpha)$,

$$C^{-1}\lambda^N \omega(B(x,r)) \leqslant \omega(B(x,\lambda r)) \leqslant C\lambda^{\mathbf{N}} \omega(B(x,r)).$$

The Dunkl transform is defined by

$$\hat{f}(x) = c_h \int_{\mathbb{R}^N} E(x, -iy) f(y) d\omega(y),$$

where the usual character $e^{-i\langle x,y\rangle}$ is replaced by $E(x, -iy) = V_\kappa(e^{-i\langle \cdot, y\rangle})(x)$ for some positive linear operator V_κ.

$$\|\hat{f}\|_2 = \|f\|_2.$$

The translation then is defined on the Dunkl transform side by

$$\widehat{\tau_y f}(x) = E(y, -ix)\hat{f}(x)$$

for all $x, y \in \mathbb{R}^N$.
For $f, g \in L^2(\mathbb{R}^N, h_\kappa^2)$, their convolution can be defined in terms of the translation operator by

$$f *_\kappa g(x) = \int_{\mathbb{R}^N} f(y)\tau_x g^\vee(y) h_\kappa^2(y) dy,$$

where $g^\vee(y) = g(-y)$.

The Dunkl operators

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha \in R^+} \frac{\kappa(\alpha)}{2} \langle \alpha, e_j \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}.$$

The Dunkl Laplacian

$$\triangle f(x) = \sum_{j=1}^{N} T_j^2 f(x) = \triangle_{\mathbb{R}^N} f(x) + \sum_{\alpha \in R} \kappa(\alpha) \delta_\alpha f(x),$$

where

$$\delta_\alpha f(x) = \frac{\partial_\alpha f(x)}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2}.$$

The heat semigroup

$$H_t f(x) = e^{t\triangle} f(x) = \int_{\mathbb{R}^N} h_t(x,y) f(y) d\omega(y),$$

where the heat kernel $h_t(x,y)$ is a C^∞ function for all $t > 0, x, y \in \mathbb{R}^N$ and satisfies $h_t(x,y) = h_t(y,x) > 0$ and $\int_{\mathbb{R}^N} h_t(x,y) d\omega(y) = 1$. The Dunkl-Riesz trasforms for $j = 1, 2, ..., N$,

$$R_j(f) = -T_j(\triangle)^{-1/2} f = -c \int_0^\infty T_j e^{t\triangle} f \frac{dt}{\sqrt{t}}$$

$$\widehat{(R_j(f))}(\xi) = -i \frac{\xi_j}{||\xi||} \widehat{f}(\xi).$$

The Poisson semigroup is given by

$$P_t f(x) = \pi^{-\frac{1}{2}} \int_0^\infty e^{-u} exp\Big(\frac{t^2}{4u}\triangle\Big) f(x) \frac{du}{u^{\frac{1}{2}}}$$

and $u(x,t) = P_t f(x)$, so-called the Dunkl Poisson integral, solves the boundary value problem in the half-space $\mathbb{R}^N_+$.

$$\begin{cases} (\partial_t^2 + \triangle_x) u(x,t) = 0, \\ u(x,0) = f(x). \end{cases}$$

$$|\partial_t^m \partial_x^\alpha \partial_y^\beta p_t(x,y)| \lesssim t^{-m-|\alpha|-|\beta|} \frac{1}{V(x,y,t+d(x,y))} \frac{t}{t+\|x-y\|}$$

where $V(x,y,t+d(x,y)) = V(x,t+d(x,y)) + V(y,t+d(x,y))$.

The basic geometry in the Dunkl setting

$$\|x-y\| = \{\sum_{j=1}^{N} |x_j - y_j|^2\}^{\frac{1}{2}}$$

and

$$B(x,r) =: \{y \in \mathbb{R}^N : \|x-y\| < r\}.$$

The Dunkl metric

$$d(x,y) = \min_{\sigma \in G} \|x - \sigma(y)\|.$$

and

$$B_d(x,r) =: \{y \in \mathbb{R}^N : d(x,y) < r\}.$$

$$d(x,y) \leq \|x-y\|, d(x,y) \leq d(x,z) + (z,y)$$

$$\omega(B(x,r)) \leq \omega(B_d(x,r)) \leq |G|\omega(B(x,r)).$$

Definition

An operator $T : C_0^\eta(\mathbb{R}^N) \to (C_0^\eta(\mathbb{R}^N))'$ for any $\eta > 0$, is said to be a singular integral operator if $K(x,y)$, the kernel of T, satisfies the following conditions: for some $0 < \varepsilon \leq 1$,

(i) $|K(x,y)| \lesssim \frac{1}{\omega(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^\varepsilon, x \neq y$;

(ii) $|K(x,y) - K(x,y')| \lesssim \left(\frac{\|y-y'\|}{\|x-y\|}\right)^\varepsilon \frac{1}{\omega(B(x,d(x,y)))}, \|y-y'\| \leq d(x,y)/2$;

$|K(x',y) - K(x,y)| \lesssim \left(\frac{\|x-x'\|}{\|x-y\|}\right)^\varepsilon \frac{1}{\omega(B(x,d(x,y)))}, \|x-x'\| \leq d(x,y)/2$.

(iii) $\langle T(f), g\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)f(x)g(y)d\omega(x)d\omega(y)$ for $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$.

(i) $\omega(x, \|x-y\|) = \omega(x, \frac{\|x-y\|}{d(x,y)} d(x,y)) \geq C\left(\frac{\|x-y\|}{d(x,y)}\right)^N \omega(x, d(x,y))$

(ii) $\frac{1}{\omega(x,\|x-y\|)} \lesssim \frac{1}{\omega(x,d(x,y))}\left(\frac{d(x,y)}{\|x-y\|}\right)^N \lesssim \frac{1}{\omega(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^\varepsilon$

(iii) $|K(x,y) - K(x,y')| \lesssim \left(\frac{\|y-y'\|}{\|x-y\|}\right)^\varepsilon \frac{1}{\omega(B(x,d(x,y)))} =$

$\left(\frac{\|y-y'\|}{d(x,y)}\right)^\varepsilon \frac{1}{\omega(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^\varepsilon$, for $\|y - y'\| \leq d(x,y)/2$.

(iv) $\int_{\delta<\|x-y\|<R} |K(x,y)| dw(y) \leq$
$C\frac{1}{\delta^\varepsilon} \int_{d(x,y)<R} \frac{d(x,y)^\varepsilon}{\omega(B(x,d(x,y)))} dw(y) \leq C\frac{R^\varepsilon}{\delta^\varepsilon} < \infty.$

Theorem

Suppose that T is a Dunkl-Calderón-Zygmund singular integral operator. Then T extends to a bounded operator on $L^2(\mathbb{R}^N, \omega)$ if and only if (a) $T(1)(x) \in BMO_{cw}(\mathbb{R}^N, \omega)$; (b) $T^*(1)(x) \in BMO_{cw}(\mathbb{R}^N, \omega)$; (c) $T \in WBP$.

Almost orthogonal estimate: if $T(1) = T^*(1) = 0$ and $T \in WBP$:

$$\|D_k^* T D_{k'}\|_{L^2(\omega) \mapsto L^2(\omega)} \lesssim 2^{-|k-k'|}.$$

Definition

A function $f(x)$ is said to be a smooth molecule for $0 < \beta \leq 1, \gamma > 0, r > 0$ and some fixed $x_0 \in \mathbb{R}^N$, if $f(x)$ satisfies the following conditions:

(i) $|f(x)| \leq C \frac{1}{V(x,x_0,r+d(x,x_0))}\left(\frac{r}{r+\|x-x_0\|}\right)^{\gamma}$;

(ii) $|f(x)-f(x')| \leq C\left(\frac{\|x-x'\|}{r}\right)^{\beta}\Big\{\frac{1}{V(x,x_0,r+d(x,x_0))}\left(\frac{r}{r+\|x-x_0\|}\right)^{\gamma}$
$+\frac{1}{V(x',x_0,r+d(x',x_0))}\left(\frac{r}{r+\|x'-x_0\|}\right)^{\gamma}\Big\}$;

(iii) $\int_{\mathbb{R}^N} f(x)d\omega(x) = 0$.

If $f(x)$ is a smooth molecule, denote $f(x)$ by $f \in \mathbb{M}(\beta, \gamma, r, x_0)$ and define the norm of f by

$$\|f\|_{\mathbb{M}(\beta,\gamma,r,x_0)} =: \inf\{C\}.$$

Definition

A function $f(x)$ is said to be a weak smooth molecule for $0 < \beta \leq 1, \gamma > 0, r > 0$ and some fixed $x_0 \in \mathbb{R}^N$, if $f(x)$ satisfies the following conditions:

(i) $|f(x)| \leq C \frac{1}{V(x,x_0,r+d(x,x_0))}\left(\frac{r}{r+d(x,x_0)}\right)^{\gamma}$;

(ii) $|f(x) - f(x')| \leq C\left(\frac{\|x-x'\|}{r}\right)^{\beta}\Big\{\frac{1}{V(x,x_0,r+d(x,x_0))}\left(\frac{r}{r+d(x,x_0)}\right)^{\gamma} + \frac{1}{V(x',x_0,r+d(x',x_0))}\left(\frac{r}{r+d(x',x_0)}\right)^{\gamma}\Big\}$;

(iii) $\int_{\mathbb{R}^N} f(x)d\omega(x) = 0$.

$$\|f\|_{\widetilde{\mathbb{M}}(\beta,\gamma,r,x_0)} =: \inf\{C\}.$$

Theorem

Suppose that T is the Dunkl-Calderón-Zygmund singular integral operator with $T(1) = T^*(1) = 0$ and $T \in WBP$. Then T maps $\mathbb{M}(\beta, \gamma, r, x_0)$ to $\widetilde{\mathbb{M}}(\beta', \gamma', r, x_0)$ with $0 < \beta' < \beta < \varepsilon$ and $0 < \gamma' < \gamma < \varepsilon$, where ε is the exponent of the regularity of the kernel of T. Moreover, there exists a constant C such that

$$\|T(f)\|_{\widetilde{\mathbb{M}}(\beta', \gamma', r, x_0)} \leq C \|f\|_{\mathbb{M}(\beta, \gamma, r, x_0)}.$$

Lemma

Let $x, y \in \mathbb{R}^N$ and $t, s > 0$ with $t \geq s$. Suppose that $f_t(x, \cdot)$ is a weak smooth molecule function in $\tilde{\mathbb{M}}(1, 1, t, x)$ and $g_s(\cdot, y)$ is a smooth molecule function in $\mathbb{M}(1, 1, s, y)$. Then for any $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists $C > 0$ such that for all $t \geq s > 0$,

$$\int_{\mathbb{R}^N} f_t(x, u) g_s(u, y) d\omega(u) \leq C \left(\frac{s}{t}\right)^{\varepsilon_1} \frac{1}{V(x,y,t+d(x,y))} \left(\frac{t}{t+d(x,y)}\right)^{\varepsilon_2}.$$

Lemma

Let T be a Dunkl-Calderón-Zygmund singular integral satisfying $T1 = T^*1 = 0$ and $T \in WBP(d\omega)$. Then

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} D_k(x, u) K(u, v) D_j(v, y) d\omega(u) d\omega(v) \right|$$

$$\lesssim r^{-|k-j|\varepsilon'} \frac{1}{V(x,y,r^{-j\vee -k}+d(x,y))} \left(\frac{r^{-j\vee -k}}{r^{-j\vee -k}+d(x,y)}\right)^{\varepsilon'}.$$

The Calderón reproducing formula [Anker, Dziubański and Hejna, 2019]:
For $f \in L^2(\mathbb{R}^N, \omega)$,

$$f(x) = \int_0^\infty \psi_t * q_t * f(x) \frac{dt}{t},$$

where $q_t = t\partial_t p_t$ with the Poisson kernel p_t, $q_t * f(x) = \int_{\mathbb{R}^N} q_t(x,y) f(y) d\omega(y)$ and $\psi_t * f(x) = \int_{\mathbb{R}^N} \psi_t(x,y) f(y) d\omega(y)$ with $\psi(x)$ being a radial Shwartz function supported in the unit ball $B(0,1)$.

$$f(x) = \int_0^\infty \psi_t * q_t * f(x)\frac{dt}{t} - \sum_{j=-\infty}^{\infty} \int_{r^{-j}}^{r^{-j+1}} \psi_j * q_j * f(x)\frac{dt}{t}$$

$$+ \sum_{j=-\infty}^{\infty} \int_{r^{-j}}^{r^{-j+1}} \left[\psi_j * q_j f(x) - \psi_t * q_t f(x)\right]\frac{dt}{t}$$

$$= -\ln r \sum_{j=-\infty}^{\infty} \psi_j * q_j * f(x) + R_1(f)$$

$$-\ln r \sum_{j=-\infty}^{\infty} \psi_j * q_j * f(x) = -\ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \int_Q \psi_j(x,y) q_j * f(y) d\omega(y)$$

$$Q^j = \{Q, l(Q) = 2^{-M-j}\}$$

$$= -\ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} w(Q)\psi_j(x, x_Q) q_j * f(x_Q)$$

$$+ \ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \int_Q \Big[\psi_j(x, x_Q) q_j * f(x_Q) - \psi_t(x, y) q_j * f(y)\Big] d\omega(y)$$

$$= T_M(f)(x) + R_M f(x)$$

$$T_M(f)(x) = -\ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} w(Q)\psi_j(x, x_Q) q_j * f(x_Q)$$

$$I = T_M + R_1 + R_M$$

R_1 and R_M are Dunkl-Calderón-Zygmund operators.
For $1 < p < \infty$,

$$||R_1(f)||_p \leq C(r-1)||f||_p$$

and

$$||R_M(f)||_{\leq} r^{-M}||f||_p.$$

The wevelet–type decomposition:

$$f(x) = T_M(T_M)^{-1}(f)(x) =$$

$$-\ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} w(Q)\psi_j(\cdot, x_Q) q_j * (T_M)^{-1}(f)(x_Q).$$

Theorem

If $f \in L^2 \cap L^p(\mathbb{R}^N, \omega), 1 < p < \infty$, and x_Q are any fixed point in the cubes Q, then there exists the function $h \in L^2 \cap L^p(\mathbb{R}^N, \omega)$, such that $||f||_p \sim ||h||_p$,

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \omega(Q)\psi_j(x, x_Q) q_j * h(x_Q),$$

where the series converges in $L^2 \cap L^p(\mathbb{R}^N, \omega)$.

Definition

For $f \in L^2(\mathbb{R}^N, \omega)$, $S(f)$, the *discrete Littlewood–Paley square function* of f, is defined by

$$S(f)(x) := \Big\{ \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} |q_Q f(x_Q)|^2 \chi_Q(x) \Big\}^{1/2},$$

where $q_Q = q_j$ when $Q \in Q^j$ and $q_t = t\frac{\partial}{\partial t} p_t$ with p_t is the Poisson kernel, and $\chi_Q(x)$ is the characteristical function of the cube Q.

Theorem

There exist two constants C and C' such that for $L^p(\mathbb{R}^N, \omega), 1 < p < \infty,$

$$C' \|f\||_p \leq \|S(f)\|_p \leq C\|f\|_p.$$

Definition

For $f \in L^2(\mathbb{R}^N, \omega)$, $||f||_{H^p_d}$, the Dunkl-Hardy space norm of f, is defined by $||f||_{H^p_d} := ||S(f)||_p$ for $0 < p \leq 1$.

$$I = T_M + R_1 + R_M.$$

For $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leq 1$ and $f \in L^2(\mathbb{R}^N, d\omega)$,

$$||R_1(f)||_{H^p_d} \leq C(r-1)||f||_{H^p_d}$$

and

$$||R_M(f))||_{H^p} \leq Cr^{-M}||f||_{H^p}$$

For $f \in L^2(\mathbb{R}^N, d\omega)$,

$$f(x) = T_M(T_M)^{-1}(f)(x) =$$

$$-\ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} w(Q)\psi_j(\cdot, x_Q)q_j * ((T_M)^{-1}(f)(x_Q)).$$

Theorem

If $f \in L^2(\mathbb{R}^N, \omega)$ with $||f||_{H_d^p} < \infty$, for $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leq 1$, then there exists the function $h \in L^2(dw)$, such that $||f||_2 \sim ||h||_2, ||f||_{H_d^p} \sim ||h||_{H_d^p}$ and

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \omega(Q) \psi_Q(x, x_Q) q_Q h(x_Q),$$

where the series converges in $L^2(d\omega)$ norm and the Dunkl-Hardy space norm.

The range $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leq 1$ is sharp and this is the same as the classical case.

Theorem

For $f, g \in L^2(\mathbb{R}^N, \omega)$ and $\frac{\mathbf{N}}{\mathbf{N+1}} < p \le 1$, then there exists a constant C such that

$$|\langle f, g\rangle| \le C||f||_{H_d^p}||g||_{CMO_d^p}.$$

Definition

Suppose that $f \in L^2(\mathbb{R}^N, \omega)$ and $0 < p < \infty$. The norm of $f \in CMO_d^p(\mathbb{R}^N, \omega)$ is defined by

$$\|f\|_{CMO_d^p(\omega)} =: \sup_P \left\{ \frac{1}{\omega(P)^{\frac{2}{p}-1}} \sum_{Q \subseteq P} \omega(Q) \left|\psi_Q * f(x_Q)\right|^2 \right\}^{1/2} < \infty,$$

where P runs over all dyadic cubes and $\psi_Q = \psi_j$ when $Q \in Q^j$.

$$I = T_M + R_1 + R_M.$$

For $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leq 1$ and $f \in L^2(\mathbb{R}^N, d\omega)$,

$$||R_1(f)||_{CMO^p} \leq C(r-1)||f||_{CMO^p}$$

and

$$||R_M(f))||_{CMO^p} \leq Cr^{-M}||f||_{CMO^p}.$$

For $f \in L^2(\mathbb{R}^N, d\omega)$,

$$f(x) = T_M(T_M)^{-1}(f)(x) =$$

$$-\ln r \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} w(Q)\psi_j(\cdot, x_Q)q_j * ((T_M)^{-1}(f))(x_Q).$$

Theorem

If $f \in L^2(\mathbb{R}^N, \omega)$ *with* $\|f\|_{CMO_d^p} < \infty$ *for* $\frac{\mathbf{N}}{\mathbf{N+1}} < p \leqslant 1$, *then there exists a function* $h \in L^2(\mathbb{R}^N, \omega)$ *such that* $\|f\|_2 \sim \|h\|_2, \|f\|_{CMO_d^p} \sim \|h\|_{CMO_d^p}$ *and for each* $g \in L^2 \cap H_d^p(\mathbb{R}^N, \omega)$,

$$\langle f, g \rangle = \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \omega(Q) \psi_Q g(x_Q) q_Q h(x_Q)$$

where the last series converges absolutely.

The range $\frac{\mathbf{N}}{\mathbf{N+1}} < p \leq 1$ is sharp and this is the same as the classical case.

Theorem

Let $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leq 1$. *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence in* $L^2(\mathbb{R}^N, \omega)$ *with* $\|S(f_n - f_m)\|_p \to 0$ *as* $n, m \to \infty$. *Then there exists* $f \in (L^2 \cap CMO_d^p)'$, *such that*

$$\|S(f)\|_p = \lim_{n\to\infty} \|S(f_n)\|_p < \infty;$$

There exists a distribution $h \in (L^2 \cap CMO_d^p)'$ *with* $\|S(f)\|_p \sim \|S(h)\|_p$ *such that for each* $g \in L^2 \cap CMO_d^p(\mathbb{R}^N, \omega)$, *the following weak-type discrete Calderón reproducing formula holds in the distribution sense:*

$$\langle f, g\rangle = \sum_{j=-\infty}^{\infty} \sum_{Q\in Q^j} w(Q)\psi_Q g(x_Q) q_Q h(x_Q),$$

where the last serise converges absolutely.

Theorem

Let $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leq 1$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^2(\mathbb{R}^N, \omega)$ with $\|f_n - f_m\|_{CMO_d^p} \to 0$ as $n, m \to \infty$. Then there exists $f \in (L^2 \cap H_d^p(\mathbb{R}^N, \omega))'$, such that

$$\|f\|_{CMO_d^p} = \lim_{n\to\infty} \|f_n\|_{CMO_d^p} < \infty;$$

There exists a distribution $h \in (L^2 \cap H_d^p(\mathbb{R}^N, \omega))'$ with $\|f\|_{CMO_d^p} \sim \|h\|_{CMO_d^p}$, such that for each $g \in L^2 \cap H_d^p(\mathbb{R}^N, \omega)$, the following weak-type discrete Calderón reproducing formula holds in the distribution sense:

$$\langle f, g\rangle = \sum_{j=-\infty}^{\infty} \sum_{Q\in Q^j} w(Q)\psi_Q g(x_Q) q_Q h(x_Q),$$

where the last series converges absolutely.

Definition

The Dunkl-Hardy space $H_d^p(\mathbb{R}^N, \omega)$, $\frac{\mathbf{N}}{\mathbf{N+1}} < p \leq 1$, is defined by the collection of all distributions $f \in (L^2(\mathbb{R}^N, \omega) \cap CMO_d^p(\mathbb{R}^N, \omega))'$ such that

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \omega(Q)\lambda_Q \psi_Q(x, x_Q)$$

with

$$\|\Big\{ \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} |\lambda_Q|^2 \chi_Q \Big\}^{1/2}\|_p < \infty,$$

where the series converges in the distribution sense.

$$\|f\|_{H_d^p} =: \inf \Big\{ \|\Big\{ \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} |\lambda_Q|^2 \chi_Q(x) \Big\}^{1/2}\|_p \Big\},$$

where the infimum is taken over all $f(x) = \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \omega(Q)\lambda_Q \psi_Q(x, x_Q)$.

Definition

The Dunkl–Carleson measure space $CMO_d^p(\mathbb{R}^N, \omega)$, $\frac{\mathbf{N}}{\mathbf{N+1}} < p \leqslant 1$, is defined as the set of all distributions $f \in (L^2 \cap H_d^p(\mathbb{R}^N, \omega))'$ represented by $f(x) = \sum_{j=-\infty}^{\infty} \sum_{Q \in Q^j} \omega(Q)\lambda_Q \psi_Q(x, x_Q)$ with

$$\sup_P \left\{ \frac{1}{\omega(P)^{\frac{2}{p}-1}} \sum_{Q \subseteq P} \omega(Q)\left|\lambda_Q\right|^2 \right\}^{1/2} < \infty,$$

where the series converges in the distribution sense. If $f \in CMO_d^p(\mathbb{R}^N, \omega)$ the norm of f is defined by

$$\|f\|_{CMO_d^p} := \inf \left\{ \sup_P \left\{ \frac{1}{\omega(P)^{\frac{2}{p}-1}} \sum_{Q \subseteq P} \omega(Q)\left|\lambda_Q\right|^2 \right\}^{1/2} \right\},$$

where the infimum is taken over all f represented as above.

Theorem

Let $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leqslant 1$. *Then*

$$H^p_d(\mathbb{R}^N, \omega) = \overline{L^2 \cap H^p_d(\mathbb{R}^N, \omega)},$$

where $\overline{L^2 \cap H^p_d(\mathbb{R}^N, \omega)}$ *is the collection of all distributions* $f \in (L^2 \cap CMO^p_d(\mathbb{R}^N, \omega))'$ *such that there exists a sequence* $\{f_n\}_{n=1}^\infty$ *in* $L^2(\mathbb{R}^N, \omega)$ *with* $\|f_n\|_{H^p_d} \leqslant C\|f\|_{H^p_d}$ *for all* n *and* f_n *converges to* f *in* $(L^2 \cap CMO^p_d(\mathbb{R}^N, \omega))'$. *Moreover,* $\|f_n - f\|_{H^p_d} \to 0$ *as* $n \to \infty$.

$$CMO^p_d(\mathbb{R}^N, \omega) = \overline{L^2 \cap CMO^p_d(\mathbb{R}^N, \omega)},$$

where $\overline{L^2 \cap CMO^p_d(\mathbb{R}^N, \omega)}$ *is the collection of all distributions* $f \in (L^2 \cap H^p_d(\mathbb{R}^N, \omega))'$ *such that there exists a sequence* $\{f_n\}_{n=1}^\infty$ *in* $L^2(\mathbb{R}^N, \omega)$ *with* $\|f_n\|_{CMO^p_d} \leqslant C\|f\|_{CMO^p_d}$ *for all* n *and* f_n *converges to* f *in* $(L^2 \cap H^p_d(\mathbb{R}^N, \omega))'$. *Moreover,* $\langle f_n - f_m, g\rangle \to 0$ *as* $n, m \to \infty$ *for all* $g \in L^2 \cap H^p_d(\mathbb{R}^N, \omega)$.

Theorem

Let $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leqslant 1$. *Then*

$$\Big(H_d^p(\mathbb{R}^N,\omega)\Big)' = CMO_d^p(\mathbb{R}^N,\omega).$$

Theorem

Suppose $\frac{\mathbf{N}}{\mathbf{N+1}} < p \leqslant 1$. *If* $f \in H_d^p(\mathbb{R}^N, \omega)$ *then* f *has an atomic decomposition. More precisely,* $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$, *where all* a_j *are* $(2, p)$ *atoms, the series converges in* $(L^2 \cap CMO_d^p(\mathbb{R}^N, \omega))'$ *and for some constant* C,

$$\sum_{j=-\infty}^{\infty} |\lambda_j|^p \leqslant C\|f\|_{H_d^p}^p.$$

Conversely, if f *has an atomic decomposition* $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ *in the sense* $(L^2 \cap CMO_d^p(\mathbb{R}^N, \omega))'$, *then* $f \in H_d^p(\mathbb{R}^N, \omega)$ *and*

$$\|f\|_{H_d^p}^p \leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

Theorem

Suppose that T is a bounded operator on $L^2(\mathbb{R}^N,\omega)$ with the kernel $K(x,y)$ satisfies the smoothness conditions only for $0<\varepsilon\leqslant 1$ *and* $\|y-y'\|\leqslant d(x,y)/2$,

$$|K(x,y)-K(x,y')|\leqslant C\Big(\frac{\|y-y'\|}{\|x-y\|}\Big)^{\varepsilon}\frac{1}{\omega(B(x,d(x,y)))}.$$

Then T is bounded from the Dunkl–Hardy space $H^p_d(\mathbb{R}^N,\omega)$ to $L^p(\mathbb{R}^N,\omega)$ for $\frac{\mathbf{N}}{\mathbf{N}+\varepsilon}<p\leqslant 1$.
When $p=1$, the above smoothness condition can be replaced by the following Hörmander condition:

$$\int_{\|y-y'\|\leqslant d(x,y)/2}|K(x,y)-K(x,y')|d\omega(x)\leqslant C$$

and T is also bounded from $H^1_d(\mathbb{R}^N,\omega)$ to $L^1(\mathbb{R}^N,\omega)$.

Definition

Suppose $\frac{\mathbf{N}}{\mathbf{N+1}} < p \leqslant 1$. A function $m(x) \in L^2(\mathbb{R}^N, \omega)$ is said to be an $(p, 2, \varepsilon, \eta)$ molecule centered at $x_0 \in \mathbb{R}^N$ for the Dunkl–Hardy space $H_d^p(\mathbb{R}^N, \omega)$ if $1 \geqslant \varepsilon > \eta > 0$, $\frac{\mathbf{N}}{\mathbf{N}+\varepsilon-\eta} < p \leqslant 1$, $\int_{\mathbb{R}^N} m(x) d\omega(x) = 0$ and

$$\Big(\int_{\mathbb{R}^N} m(x)^2 d\omega(x)\Big) \times$$

$$\Big(\int_{\mathbb{R}^N} m(x)^2 \omega(B(x_0, \|x - x_0\|))^{1+\frac{2\varepsilon-2\eta}{\mathbf{N}}} d\omega(x)\Big)^{(\frac{\mathbf{N}+2\varepsilon-2\eta}{\mathbf{N}} \frac{p}{2-p} - 1)^{-1}} \leqslant 1.$$

Theorem

Suppose that $\frac{\mathbf{N}}{\mathbf{N}+\varepsilon-\eta} < p \leqslant 1$ *and* m *is an* $(p, 2, \varepsilon, \eta)$ *molecule. Then* $m \in H^p_d(\mathbb{R}^N, \omega)$ *and moreover,*

$$\|m\|_{H^p_d} \leqslant C,$$

where the constant C *is independent of* m.

Theorem

Suppose that T *is a bounded operator on* $L^2(\mathbb{R}^N, \omega)$ *with the kernel* $K(x, y)$ *satisfying the following smoothness condition when* $M > \frac{\mathbf{N}}{2}, 0 < \varepsilon \leqslant 1$ *and* $\|y - y'\| \leqslant \frac{1}{2} d(x, y)$,

$$|K(x, y) - K(x, y')| \leqslant C\left(\frac{\|y - y'\|}{\|x - y\|}\right)^{\varepsilon} \frac{1}{\omega(B(x, d(x, y)))} \left(\frac{d(x, y)}{\|x - y\|}\right)^{M}.$$

Then T *is bounded on the Dunkl–Hardy space* $H_d^p(\mathbb{R}^N, \omega)$, $\frac{\mathbf{N}}{\mathbf{N}+\varepsilon} < p \leqslant 1$, *if and only if* $T^*(1) = 0$.

Theorem

Suppose that $\frac{\mathbf{N}}{\mathbf{N}+\varepsilon} < p \leqslant 1$ *and* T *is a Dunkl–Calderón–Zygmund operator with the exponent of the regularity of the kernel* ε. *Then* T *is bounded on* $H^p_d(\mathbb{R}^N, \omega)$ *if and only if* $T^*(1) = 0$ *and* T *is bounded on* $CMO^p_d(\mathbb{R}^N, \omega)$ *if and only if* $T(1) = 0$.

Theorem

Let $\frac{\mathbf{N}}{\mathbf{N}+1} < p \leqslant 1$. *The Dunkl–Riesz transforms* $R_j, 1 \leqslant j \leqslant N$, *are bounded on the Hardy space* $H^p_d(\mathbb{R}^N, \omega)$, *from* $H^p_d(\mathbb{R}^N, \omega)$ *to* $L^p(\mathbb{R}^N, \omega)$ *and on the Dunkl–Carleson measure space* $CMO^p_d(\mathbb{R}^N, \omega)$.

Thank You!