Approximation of images by group invariant subspaces

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Joint work with D. Barbieri, C. Cabrelli and U. Molter

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Let $\Lambda \subset \mathbb{Z}_d \times \mathbb{Z}_d$ be a sublattice $\Lambda = \mathbb{Z}_p \times \mathbb{Z}_p \ (p/d)$, and let $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For $\psi \in \ell_2(\mathbb{Z}_d \times \mathbb{Z}_d)$, $n = (n_1, n_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$, $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, let

$$ T(\lambda)f(n) = f(n_1 - \lambda_1, n_2 - \lambda_2), \quad R\psi(n) = \psi(rn^t). $$
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Let $V \subset \ell_2(\mathbb{Z}_d \times \mathbb{Z}_d)$ linear subspace. The error of approximation of $\mathcal{F}$ by $V$ is:

$$E[\mathcal{F}, V] = \sum_{j=1}^{m} \| f_j - P_V f_j \|^2.$$

For $\kappa \in \mathbb{N}$, let $\Psi = \{ \psi_j \}_{j=1}^{\kappa} \subset \ell_2(\mathbb{Z}_d \times \mathbb{Z}_d)$. Let $V(\Psi)$ be the invariant subspace of $\ell_2(\mathbb{Z}_d \times \mathbb{Z}_d)$ generated by $\Psi$:

$$V(\Psi) = \text{span}\left\{ T(\lambda)R^g\psi_j : \lambda \in \Lambda, g \in \{0, 1, 2, 3\}, j \in \{1, \ldots, \kappa\} \right\}.$$
**Goal**: for $\kappa \in \mathbb{N}$, find generators $\Phi = \{\phi_j\}_{j=1}^\kappa \subset \ell_2(\mathbb{Z}_d \times \mathbb{Z}_d)$ such that

$$\Phi = \arg \min \mathcal{E}[\mathcal{F}, V(\Psi)]$$

where the minima is taken over all $V(\Psi)$ with at most $\kappa$ generators. Moreover,

$$\left\{ T(\lambda)R^g \phi_j : \lambda \in \Lambda, g \in \{0, 1, 2, 3\}, j \in \{1, \ldots, \kappa\} \right\}$$

is a Parseval frame for $V(\Phi)$, that is

$$\mathbb{P}_{V(\Phi)} f = \sum_{\lambda \in \Lambda} \sum_{g=0}^3 \sum_{j=1}^\kappa \langle f, T(\lambda)R^g \phi_j \rangle T(\lambda)R^g \phi_j$$

is an orthogonal projection onto $V(\Phi)$. 
Techniques

- Translational structure on a lattice: Fourier analysis.
- Rotations: the group of symmetries $\Lambda \rtimes \{1, R, R^2, R^3\}$ is not abelian.
- Quadratic error functional: explicit solution using SVD.
The how and the why

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Motivation

- Reduce the side of the original image using the PF coefficients.
- Introduce non abelian symmetries.
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Motivation

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- Introduce non abelian symmetries.

General setting

- Functional data, defined on an LCA group.
- Symmetry: a semidirect product of a lattice of translates and a discrete group of automorphisms.
Definition of group invariance

Let $\mathbb{R}$ be an LCA group, let $\Lambda \subset \mathbb{R}$ be a lattice subgroup, and let $G \subset \text{Aut}(\mathbb{R})$ be a countable group such that $g\Lambda = \Lambda$ for all $g \in G$.

\[ T(\lambda)f(x) = f(x - \lambda), \quad R(g)f(x) = f(g^{-1}x), \] for $f \in L^2(\mathbb{R})$.

Observe that $R(g)T(\lambda) = T(g\lambda)R(g)$.

Therefore, if we consider $\Gamma = \Lambda \rtimes G = \{ (\lambda, g) : \lambda \in \Lambda, g \in G \}$, with composition law $((\lambda, g)) \cdot ((\lambda', g')) = (g\lambda' + \lambda, gg')$, $\Gamma$ acts unitarily on $L^2(\mathbb{R})$ by $T(\lambda)R(g)$.

A closed subspace $V \subset L^2(\mathbb{R})$ is $\Gamma$-invariant if $T(\lambda)R(g)V \subset V$ for all $\lambda \in \Lambda$, $g \in G$.

In this case, there is always a countable $\Psi = \{ \psi_j \}_{j \in \mathbb{N}}$ such that $V = \text{span} \{ T(\lambda)R(g)\psi_j : \lambda \in \Lambda, g \in G, j \in \mathbb{N} \}$.\[ ^1 \text{If } \mathbb{R} = \mathbb{R}^d, \text{ that is } \Lambda = AZ^d \subset \mathbb{R}^d \text{ for } A \in GL_d(\mathbb{R}). \]
Definition of group invariance

Let $\mathbb{R}$ be an LCA group, let $\Lambda \subset \mathbb{R}$ be a lattice subgroup$^1$, and let $G \subset \text{Aut}(\mathbb{R})$ be a countable group such that $g\Lambda = \Lambda$ for all $g \in G$. $\Lambda$ and $G$ act unitarily on $L^2(\mathbb{R})$ by means of the operators

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Observe that $R(g)T(\lambda) = T(g\lambda)R(g)$. Therefore, if we consider

$$\Gamma = \Lambda \rtimes G = \{(\lambda, g) : \lambda \in \Lambda, g \in G\},$$

with composition law

$$(\lambda, g) \cdot (\lambda', g') = (g\lambda' + \lambda, gg'),$$

$\Gamma$ acts unitarily on $L^2(\mathbb{R})$ by $T(\lambda)R(g)$.

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Title: Definition of group invariance

Author: Eugenio Hernández

Date: 09/10/2022
Definition of group invariance

Let $\mathbf{R}$ be an LCA group, let $\Lambda \subset \mathbf{R}$ be a lattice subgroup\(^1\), and let $G \subset \text{Aut} (\mathbf{R})$ be a countable group such that $g\Lambda = \Lambda$ for all $g \in G$. $\Lambda$ and $G$ act unitarily on $L^2(\mathbf{R})$ by means of the operators

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$\Gamma$ acts unitarily on $L^2(\mathbf{R})$ by $T(\lambda)R(g)$.

A closed subspace $\mathcal{V} \subset L^2(\mathbf{R})$ is $\Gamma$-invariant if

$$T(\lambda)R(g)\mathcal{V} \subset \mathcal{V} \quad \forall \lambda \in \Lambda, \ g \in G.$$

In this case, there is always a countable $\Psi = \{\psi_j\}_{j \in \mathbb{N}}$ such that

$$\mathcal{V} = \text{span}\left\{ T(\lambda)R(g)\psi_j : \lambda \in \Lambda, g \in G, j \in \mathbb{N} \right\}^{L^2(\mathbf{R})}.$$

\(^1\)If $\mathbf{R} = \mathbb{R}^d$, that is $\Lambda = A\mathbb{Z}^d \subset \mathbb{R}^d$ for $A \in \text{GL}_d(\mathbb{R})$. 

Eugenio Hernández  
Approximation of images  
09/10/2022
First Part

Let \( b = (1, 1) \) and \( G = \{(b^j, k) : j \in \mathbb{Z}, k \in \mathbb{Z}^2\} \).

We introduce the operation (multiplication) on \( G \):

\[
(1) \quad (b^j, m)(b^k, n) = (b^{j+k}, n + b^j \cdot m).
\]

Then \( G \), with this operation, is a group. For example,

\[
(b^j, k)^{-1} = (b^{-j}, -b^j \cdot k). \text{ This operation (1) is consistent}
\]
\((b, k)^{-1} = (b^{-1}, -b^{-1}k)\). This operation (1) is consistent with the operation on points of \(\mathcal{R}^2\) that maps \(x \in \mathcal{R}^2\) into \(b^d (x + k) \in \mathcal{R}^2\). We introduce the representation \(\pi\) of \(G\), acting on \(L^2(\mathcal{R}^2)\) defined by

\[\pi(b, k)f(x) = \int f((b, k)^{-1}x) = \int f(b^{-1}x - k)\]

for \(f \in L^2(\mathcal{R}^2)\). It is easily seen that this is a unitary (operator) representation of \(G\) acting on \(L^2(\mathcal{R}^2)\).
First Interlude: The pinwheel tiling

Cruido’s Figure. The shaded region indicates what I explained for defining $V_0$ and $V$. 
Best approximation problem

Let $\mathcal{F} = \{f_1, \ldots, f_m\} \subset L^2(\mathbb{R})$ be functional data, and let $\kappa \in \mathbb{N}$.

**Goal:** find

$$\Phi = \arg \min_{\Psi} \sum_{j=1}^{m} \| f_j - \mathbb{P}_S(\Psi) f_j \|_{L^2(\mathbb{R})}^2$$

over all $\Psi = \{\psi_j\}_{j=1}^{\kappa} \subset L^2(\mathbb{R})$, where

$$S_\Gamma(\Psi) = \text{span} \left\{ T(\lambda)R(g)\psi_j : \lambda \in \Lambda, g \in G, j = 1, \ldots, \kappa \right\}^{L^2(\mathbb{R})}.$$

**Result:** an explicit construction of $\Phi \subset S_\Gamma(\mathcal{F})$ whose $\Gamma$-orbits form a Parseval frame, i.e. for $f \in L^2(\mathbb{R})$

$$\mathbb{P}_{S_\Gamma(\Phi)} f = \sum_{\lambda \in \Lambda} \sum_{g \in G} \sum_{j=1}^{\kappa} \langle f, T(\lambda)R(g)\phi_j \rangle_{L^2(\mathbb{R})} T(\lambda)R(g)\phi_j.$$
\( \hat{\mathbb{R}} \) denotes the dual group of \( \mathbb{R} \). Duality is written as
\[
\langle \xi, x \rangle = e^{2\pi i \xi \cdot x}, \quad \xi \in \hat{\mathbb{R}}, \ x \in \mathbb{R}.
\]

The annihilator\(^2\) lattice of \( \Lambda \) is
\[
\Lambda^\perp = \{ \ell \in \hat{\mathbb{R}} : \langle \ell, k \rangle = 1 \ \forall k \in \Lambda \}.
\]

The action of \( G \) on \( \mathbb{R} \) induces an action of \( G \) on \( \hat{\mathbb{R}} \) by duality:
\[
\langle g \ast \xi, x \rangle = \langle \xi, gx \rangle \quad \xi \in \hat{\mathbb{R}}, \ x \in \mathbb{R}.
\]

\(^2\)If \( \Lambda = A\mathbb{Z}^d \) in \( \mathbb{R}^d \), then \( \Lambda^\perp = (A^t)^{-1}\mathbb{Z}^d \).
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\[
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\]

Let \( \Omega \subset \hat{\mathbb{R}} \) be such that \( |\Omega \cap (\Omega + s)| = 0 \) for \( 0 \neq s \in \Lambda^\perp \), and

\[
|\hat{\mathbb{R}} \setminus \bigcup_{s \in \Lambda^\perp} \Omega + s| = 0.
\]

\(^2\)If \( \Lambda = A\mathbb{Z}^d \) in \( \mathbb{R}^d \), then \( \Lambda^\perp = (A^t)^{-1}\mathbb{Z}^d \).
The map $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\Omega, \ell_2(\Lambda^\perp))$ is the surjective isometry

$$\mathcal{T}[f](\omega) = \{\hat{f}(\omega + s)\}_{s \in \Lambda^\perp}. $$

Observe that

$$\|\mathcal{T}[f](\omega)\|_{\ell_2(\Lambda^\perp)}^2 = \sum_{s \in \Lambda^\perp} |\hat{f}(\omega + s)|^2 := [f, f](\omega) \text{ (Bracket)}. $$
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If $\mathcal{V}$ is $\Lambda$-invariant, there exists $\Psi = \{\psi_j\}_{j \in \mathbb{N}} \subset L^2(\mathbb{R})$ such that

$$\mathcal{V} = \text{span}\{\mathcal{T}(\lambda)\psi_j : \lambda \in \Lambda, j \in \mathbb{N}\}^{L^2(\mathbb{R})}.$$
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If $\mathcal{V}$ is $\Lambda$-invariant, there exists $\Psi = \{\psi_j\}_{j \in \mathbb{N}} \subset L^2(\mathbb{R})$ such that

$$\mathcal{V} = \text{span}\{\mathcal{T}(\lambda)\psi_j : \lambda \in \Lambda, j \in \mathbb{N}\}L^2(\mathbb{R}).$$ 

The range function $J_\mathcal{V}$ of $\mathcal{V}$ is the measurable map

$$J_\mathcal{V} : \Omega \to \{\text{closed subspaces of } \ell_2(\Lambda^\perp)\}$$

given by

$$J_\mathcal{V}(\omega) = \text{span}\{\mathcal{T}[\psi_j](\omega) : j \in \mathbb{N}\}_{\ell_2(\Lambda^\perp)}.$$
We know that there exists a measurable $\hat{\mathbb{R}}/\Lambda^\perp \approx \Omega \subset \hat{\mathbb{R}}$. For the approximation problem we need more: the action of $\Lambda^\perp \rtimes G$ on $\hat{\mathbb{R}}$ must have a fundamental domain $\Omega_0 \subset \mathbb{R}$:

$$\begin{align*}
|\Omega_0 \cap g \ast \Omega_0| &\quad g \neq e \quad 0 \\
|\Omega \setminus \bigcup_{g \in G} g \ast \Omega_0| &\quad = \quad 0.
\end{align*}$$

When $b \mathbb{R}$ is connected and $G$ acts faithfully on $b \mathbb{R}$, such a fundamental domain exists if and only if $G$ is finite.
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\end{cases}
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When $\hat{\mathbb{R}}$ is connected and $G$ acts faithfully on $\hat{\mathbb{R}}$, such a fundamental domain exists if and only if $G$ is finite.
Construction of the minimizing family $\Phi = \{\phi_j\}_{j=1}^{\kappa}$ for

$$E[\Psi] = \sum_{j=1}^{m} \left\| f_j - \mathbb{P}_{S_\Gamma}(\psi) f_j \right\|_{L^2(\mathbb{R}^d)}^2.$$
Construction of the minimizing family $\Phi = \{\phi_j\}_{j=1}^\kappa$ for

$$E[\Psi] = \sum_{j=1}^m \|f_j - \mathbb{P}_{S_\Gamma(\Psi)} f_j\|^2_{L^2(\mathbb{R}^d)}.$$ 

**Step I:** equivalent expression for $E[\Psi]$:

$$E[\Psi] = \int_{\Omega_0} \sum_{j=1}^m \sum_{g \in G} \|\mathcal{T}[R(g)f_j](\omega) - \mathbb{P}_{\mathcal{J}(\omega)} \mathcal{T}[R(g)f_j](\omega)\|^2_{\ell_2(\Lambda^\perp)} d\omega$$

where $\mathcal{J}(\omega) = \text{span}\{\mathcal{T}[R(g)\psi_j](\omega) : j = 1, \ldots, \kappa, g \in G\}$ is the range function of $S_\Gamma(\Psi)$. 
Approximation by $\Gamma$-invariant spaces

Step II: For any $\omega \in \Omega_0$ consider the data

$$a_{(j,g)}(\omega) = \mathcal{T}[R(g)f_j](\omega) \in \ell_2(\Lambda^\perp) \quad j \in \{1, \ldots, m\}, g \in G.$$ 

Use SVD for the matrix $A(\omega)$ whose columns are the vectors $a_{(j,g)}(\omega)$ to find

$$\{u_{(j,g)}(\omega) : j \in \{1, \ldots, \tau\}, g \in G\}$$

that minimizes each term of the integrand in Step I (Eckart-Young).
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**Step III:** extend $\{u_{(i,g)}(\omega) : j \in \{1, \ldots, \kappa\}, g \in G\}$ from $\Omega_0$ to the whole $\Omega \approx \mathbb{R}/\Lambda^\perp$ by invariance, obtaining $\{h_{(i,g)}\} \in L^2(\Omega, \ell_2(\Delta^\perp))$. 
Step II: For any $\omega \in \Omega_0$ consider the data

$$a_{(j,g)}(\omega) = T[R(g)f_j](\omega) \in \ell_2(\Lambda^\perp) \quad j \in \{1, \ldots, m\}, g \in G.$$ 

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Step IV: pull back the solution with $T$, to

$$\phi_j = T^{-1}[h_{(j,e)}] \quad \text{and} \quad \phi_{j,g} = R_g \psi_{i,e}\psi \quad j \in \{1, \ldots, \kappa\}, g \in G.$$
TO: Faculty, Staff & Graduate Students

FROM: Ann Podleski & Guido Weiss

SUBJECT: Softball Game

The COMPLEX OUTFIELD (the grad students' co-ed softball team) has challenged the DECREPITS (the faculty no-ed softball (?) team) to a game.

The DECREPITS have accepted the challenge. The game will be held in Ozu Field at 4:30 p.m., Monday, May 12. All of you be there.

05-07-80
jsd
Dataset: 2000 natural images, grayscale (8 bits), $345 \times 345$ pixels.

**Figure:** Left: lattice of translates ($23 \times 23$ points). Right: annihilator lattice in the Fourier domain ($15 \times 15$ points).
Numerics on image-net.org 2017 dataset

Distribution on the dataset of average error by pixel $\Delta_j = \frac{\|f_j - \mathbb{P}_{S(\Phi)} f_j\|_d}{256 \ast d}$ for 8, 14 and 19 generators (reduction of the dimension to $\leq \frac{1}{7}, \frac{1}{4}$ and $\frac{1}{3}$).
Generators in the Fourier domain

**Figure:** Left: generator 1 (modulus). Right: superposition of its four rotates.
Generators in the Fourier domain

**Figure:** Left: generator 2 (modulus). Right: superposition of its four rotates.
Generators in the Fourier domain

**Figure:** Left: generator 3 (modulus). Right: superposition of its four rotates.
Generators in the Fourier domain

**Figure:** Left: generator 4 (modulus). Right: superposition of its four rotates.
Generators in the image domain

Figure: The first 6 generators.
Approximation Examples (Average error)

Figure: $K = 8$ generators (dimension $\leq \frac{1}{7}$), Error per pixel $= 6.1\%$. 
Figure: $K = 14$ generators (dimension $\leq \frac{1}{4}$), Error per pixel = 5.2%. 

Approximation Examples (Average error)
Approximation Examples (Average error)

Figure: $K = 19$ generators (dimension $\leq \frac{1}{3}$), Error per pixel = 4.6%.
Approximation Examples (High error)

Figure: $K = 8$ generators (dimension $\leq \frac{1}{7}$), Error per pixel = 9.5%.
Figure: $K = 14$ generators (dimension $\leq \frac{1}{4}$), Error per pixel = 6.9%.
Figure: \( K = 19 \) generators (dimension \( \leq \frac{1}{3} \)), Error per pixel = 5.6\%.
Figure: $K = 8$ generators (dimension $\leq \frac{1}{7}$), Error per pixel $= 2.1\%$. 
Figure: $K = 14$ generators (dimension $\leq \frac{1}{4}$), Error per pixel = 1.6%.
Approximation Examples (Low error)

Figure: $K = 19$ generators ($\text{dimension} \leq \frac{1}{3}$), Error per pixel $= 1.3\%$. 


